Derivative expansions of hadronic potentials coupled to quarks for $X(3872)$

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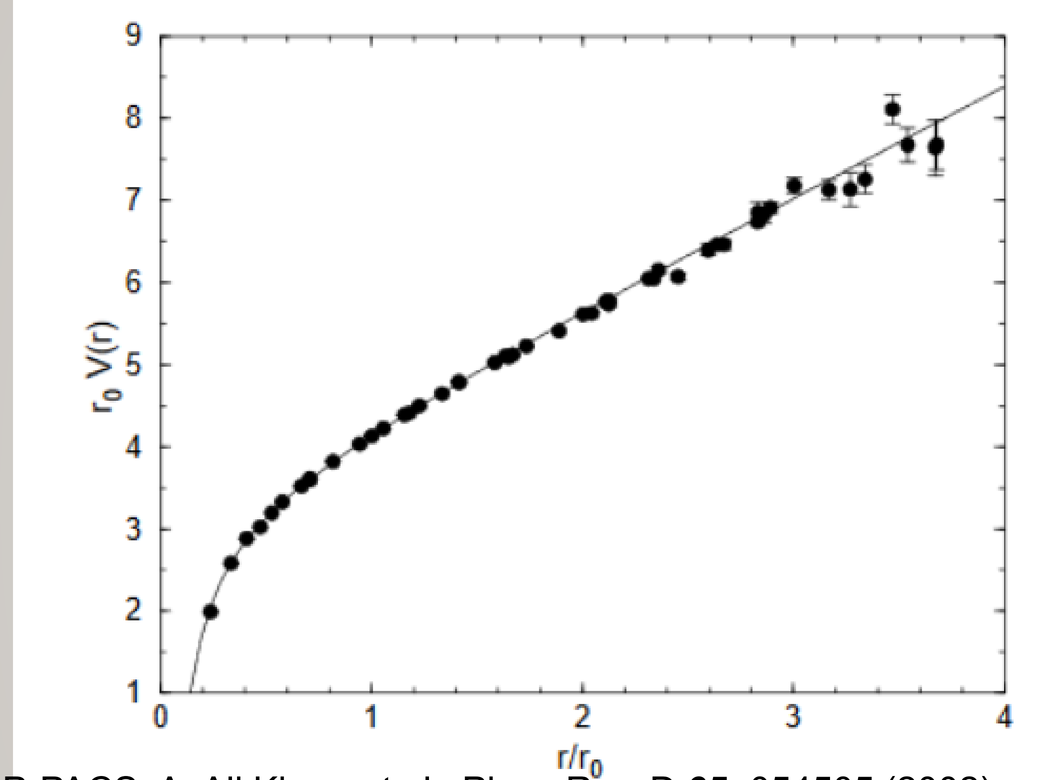
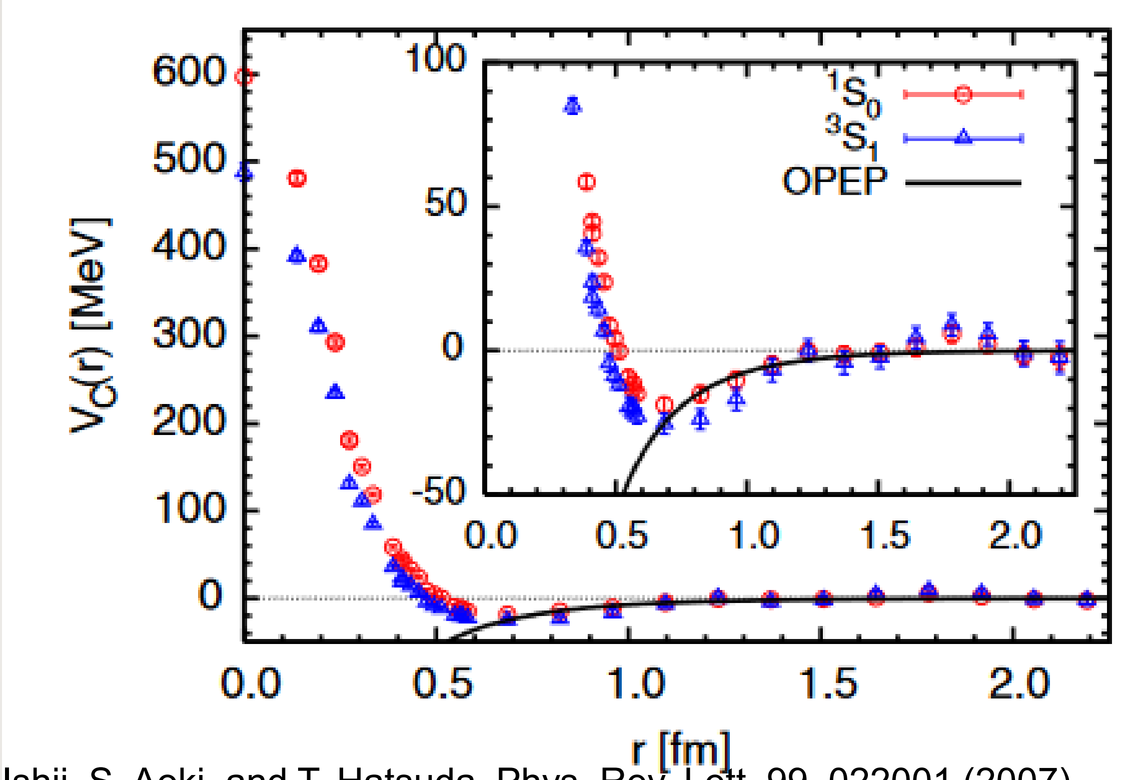
This talk is based on
[I. Terashima and T. Hyodo, PhysRevC.108.035204 (2023)]

Numerical calculation by LQCD



CP-PACS, A. Ali Khan, et al., Phys. Rev. D **65**, 054505 (2002)



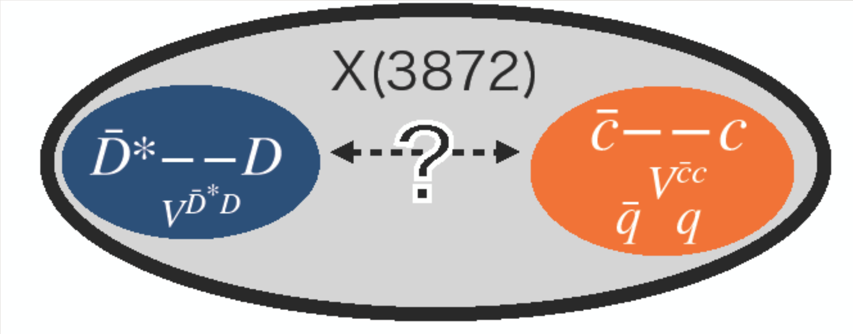
N. Ishii, S. Aoki, and T. Hatsuda, Phys. Rev. Lett. 99, 022001 (2007)

- ➢Inter-quark (static) potential
- ➢Inter-hadron (NN) potential

■Quark-antiquark potentials and hadron-hadron potentials have been studied independently

Exotic hadron $X(3872)$

- There is no restriction by QCD which prohibits the mixing with each d.o.f
  - States with same quantum numbers mix by definition
- Structure of $X(3872)$ [A. Hosaka, T. Iijima, K. Miyabayashi, Y. Sakai, and S. Yasui, PTEP **2016** (2016)]
  - **Mixing with quark** and **hadron** degrees of freedom
  - Not enough experimental data and lattice QCD results
  - How about a channel coupling between **quark** and **hadron** degrees of freedom like $\mathbf{X(3872)}$?



Goal

II-A, II-B, ... correspond to sections of Ref. [I. Terashima and T. Hyodo, PhysRevC.108.035204 (2023)]

II - A

- **Construct a framework of the channel coupling**
  - Inter-<u>hadron</u> potential with quark channel
  - Inter-<u>quark</u> potential with hadron channel

II - B

◆Local approximation

II - C

◆Analytic form (Yukawa type)

◆Applying to $X(3872)$

III - A

➢ **To study the structure of $X(3872)$,** we introduce the channel coupling

III - B

➢ **Numerical results**

Channel coupling

✓ Formulation according to Feshbach method [H. Feshbach, Ann. Phys. 5, 357 (1958); ibid., 19, 287 (1962)]

- Hamiltonian H with channel between quark potential V^q and hadron V^h

$$H = \begin{pmatrix} T^q & 0 \\ 0 & T^h + \Delta \end{pmatrix} + \begin{pmatrix} V^q & V^t \\ V^t & V^h \end{pmatrix}$$

T^q, T^h:Kinetic energy

Δ:Threshold energy

V^t:Transition potential

- Schrödinger equation with wave functions of quark and hadron channels $|q\rangle, |h\rangle$

$$H \begin{pmatrix} |q\rangle \\ |h\rangle \end{pmatrix} = E \begin{pmatrix} |q\rangle \\ |h\rangle \end{pmatrix}$$

➢ Two set of equations with quark and hadron channels are obtained

Effective potential

- Eliminate quark channel to obtain an effective Hamiltonian of hadron channel $H^h_{\mathrm{eff}}(E)$

with, $H^h_{\mathrm{eff}}(E)\,|h\rangle = E\,|h\rangle\,,$ $V^h_{\mathrm{eff}}(E)$

$$H^h_{\mathrm{eff}}(E) = T^h + \Delta^h + V^h + V^t G^q(E) V^t$$

✓ No approximation

✓ G_q is the Green function of quark channel

$$G_q(E) = (E - (T^q + V^q))^{-1}$$

➢ Quark channel contribution by coupled channels

- Coordinate representation with initial relative coordinate $\boldsymbol{r}$ and final $\boldsymbol{r}'$

$$\langle \boldsymbol{r}'_h \mid V^h_{\mathrm{eff}}(E) \mid \boldsymbol{r}_h\rangle = \langle \boldsymbol{r}'_h \mid V^h \mid \boldsymbol{r}_h\rangle + \sum_n \frac{\langle \boldsymbol{r}'_h \mid V^t \mid \phi_n\rangle \langle \phi_n \mid V^t \mid \boldsymbol{r}_h\rangle}{E - E_n}$$

➢ Quark channel contribution. Sum of discrete eigenstates E_n

◆ Energy dependent potential (denominator depends on E)
◆ Non-local potential (numerator depends on $\boldsymbol{r}, \boldsymbol{r}'$ independently)

Local approximations

✓Approximation of non-local potential to local one by two different methods

[S.Aoki and K.Yazaki, PTEP 2022, no.3, 033B04 (2022)]

① Formal derivative expansion

- Express non-local potential in terms of derivatives of delta function by **Taylor expansion** at $\boldsymbol{r} = \boldsymbol{r}'$ directly

② Derivative expansion by HAL QCD method

- Construct the potential from wave function $\psi_{k_0}(r)$ obtained from Schrödinger equation with non-local potentials at momentum k_0

➢**Solve for potentials inversely** to construct the local potentials

HAL QCD method in detail

Energy dependent

order of derivative

■Schrödinger equation with **non-local potential** at $n+1$ points of k_i $(i = 0, 1, \cdots, n)$

$$-\frac{1}{2m}\nabla^2 \psi_{k_i}(\boldsymbol{r}) + \int d^3\boldsymbol{r}' \, V_n(\boldsymbol{r}, \boldsymbol{r}', E)\psi_{k_i}(\boldsymbol{r}') = E_{k_i}\psi_{k_i}(\boldsymbol{r})$$

Unknown: $\psi_{k_i}(\boldsymbol{r})$

Obtain wavefunctions $\psi_{k_i}(\boldsymbol{r})$

Assume

■Wave functions $\psi_{k_i}(\boldsymbol{r})$ satisfy the Schrödinger equation with local potentials

$$\left(-\frac{1}{2m}\boldsymbol{\nabla}^2 + V_n(\boldsymbol{r}, \boldsymbol{\nabla})\right)\psi_{k_i}(\boldsymbol{r}) = E_{k_i}\psi_{k_i}(\boldsymbol{r}),$$

Unknown: $V_n(\boldsymbol{r}, \boldsymbol{\nabla})$

- Obtain local potential $V_n(\boldsymbol{r}, \boldsymbol{\nabla})$ by solving above equation for the potential inversely

Obtain $\psi = \psi_{k_i}$ exactly by solving local Schrödinger equation at $E = E_{k_i}$,
so that the $V_n(\boldsymbol{r}, \boldsymbol{\nabla})$ reproduces exact phase shift which is derived from $V_n(\boldsymbol{r}, \boldsymbol{r}', E)$

$X(3872)$

Construct the model of $X(3872)$

◇ Quark channel : $\bar{c}c$ ◇ Hadron channel : $D^0\ \bar{D}^{*0}$

$$\langle \phi_0 \mid V^t \mid \boldsymbol{r}_h \rangle = g_0 V(\boldsymbol{r}) = g_0 \frac{e^{-\mu r}}{r}$$

Yukawa-type form factor

g_0: coupling constant

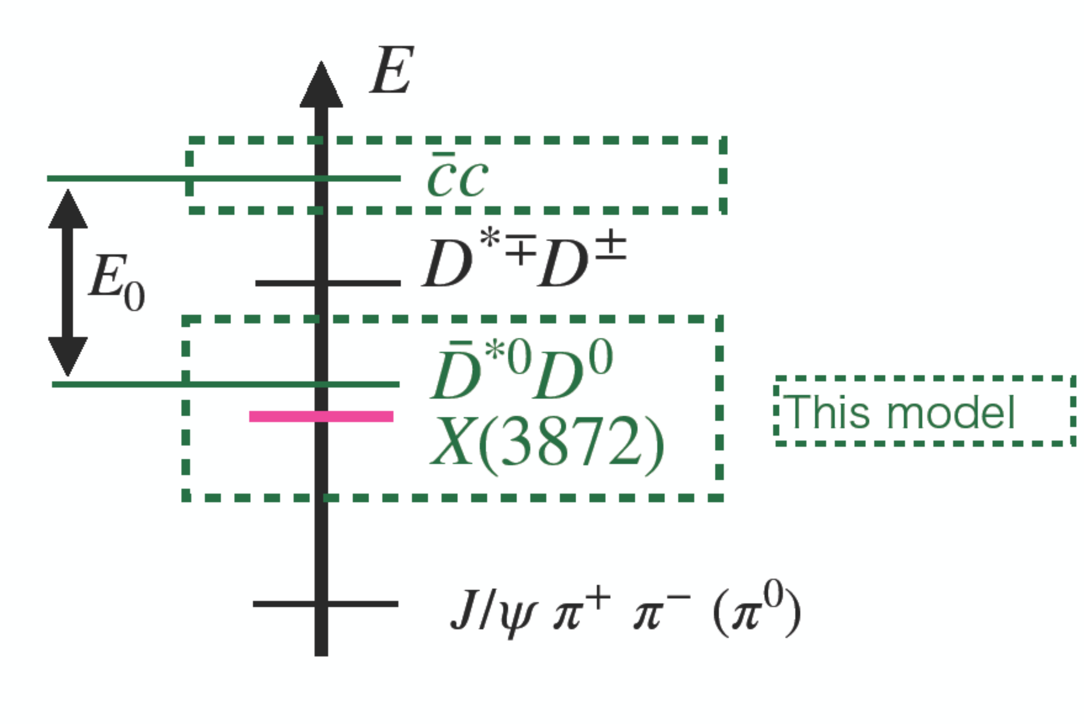
μ: cut-off

E_0: Energy relative to the threshold of the $D^0\bar{D}^{*0}$ channel

- Effective hadron potential with only $\chi_{C1}(2P)$ contribution (the strongest effect on $D^0\bar{D}^{*0}$)

$$V^h_{\text{eff}}(\boldsymbol{r}, \boldsymbol{r}', E) = \frac{g_0^2}{E - E_0} \frac{e^{-\mu r'}}{r'} \frac{e^{-\mu r}}{r}$$

- Cut-off μ is taken to be mass of π
  - ➢ Lightest exchanging meson
- Energy of $c\bar{c}$: $E_0 = m_{c\bar{c}} - (m_{D^0} + m_{\bar{D}^{*0}})$
- Coupling constant g_0 is determined to reproduce mass of $X(3872)$

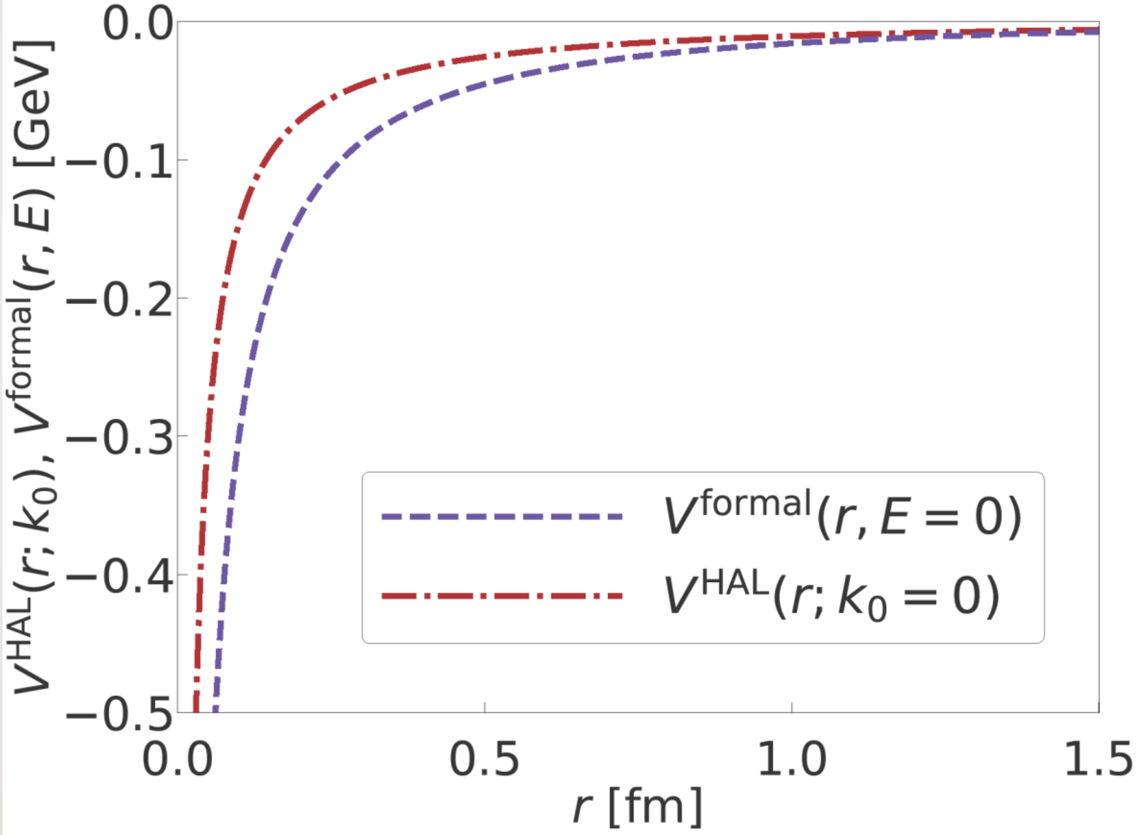


$m_{c\bar{c}}$ [S. Godfrey and N. Isgur, Phys. Rev. D, 32, 189 (1985)]

others [PDG Live]

Result : comparison of V^{HAL} and V^{formal}

- Compare approximated potentials for $X(3872)$
  - V^{HAL} and V^{formal} from the same non-local potential



- Both potentials are attractive in short-range
- Strengths of potential are quantitatively different

✓How about physical observables from these potentials?

Result : Phase sift $\delta(k)$

■Compare phase shifts $\delta(k)$ from

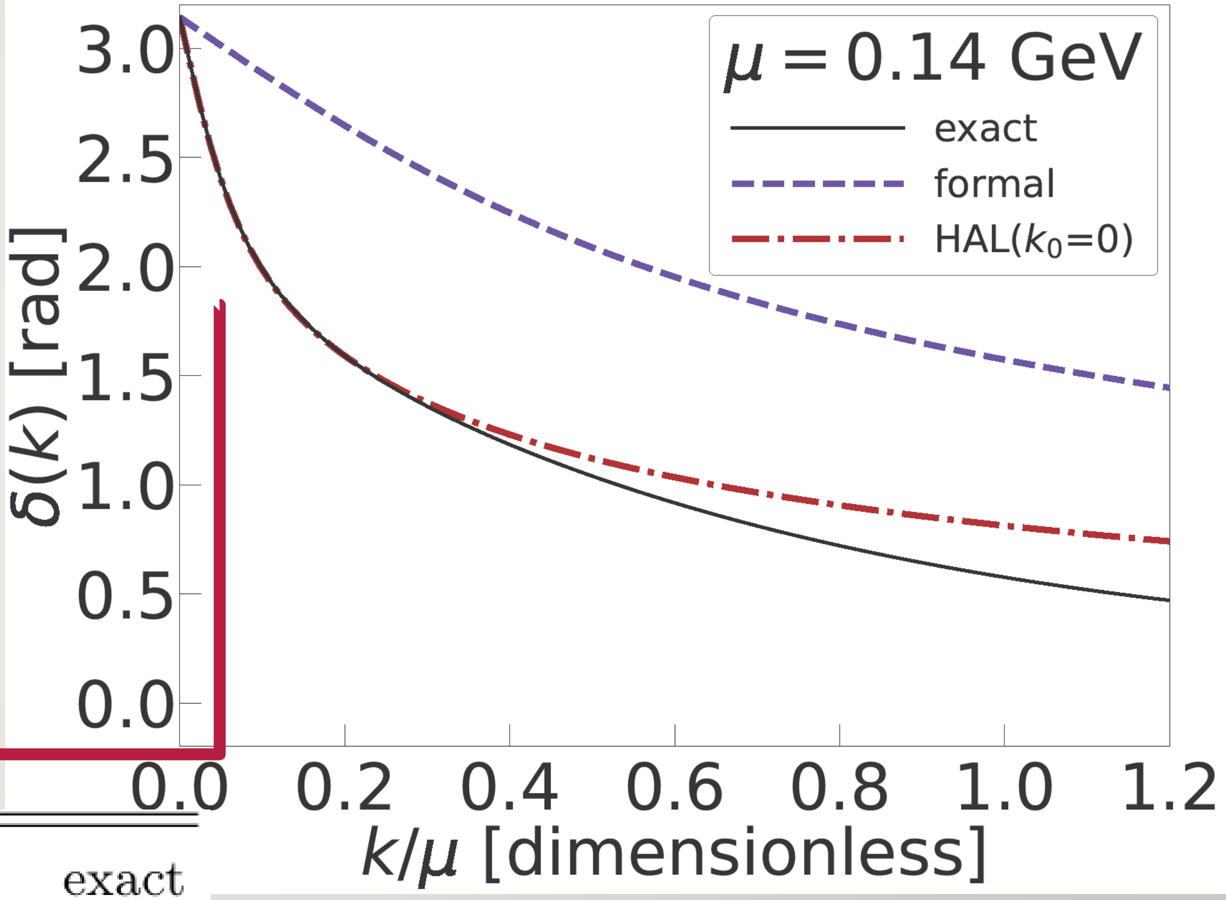
$V^{\mathrm{formal}}(r, E)$ and $V^{\mathrm{HAL}}(r; k_0 = 0)$

with exact $\delta(k)$ from non-local potential

- $\delta(k)$ from HAL QCD method reproduces exact $\delta(k)$, especially for small k

Scattering lengths are obtained from the gradient at $k = 0$

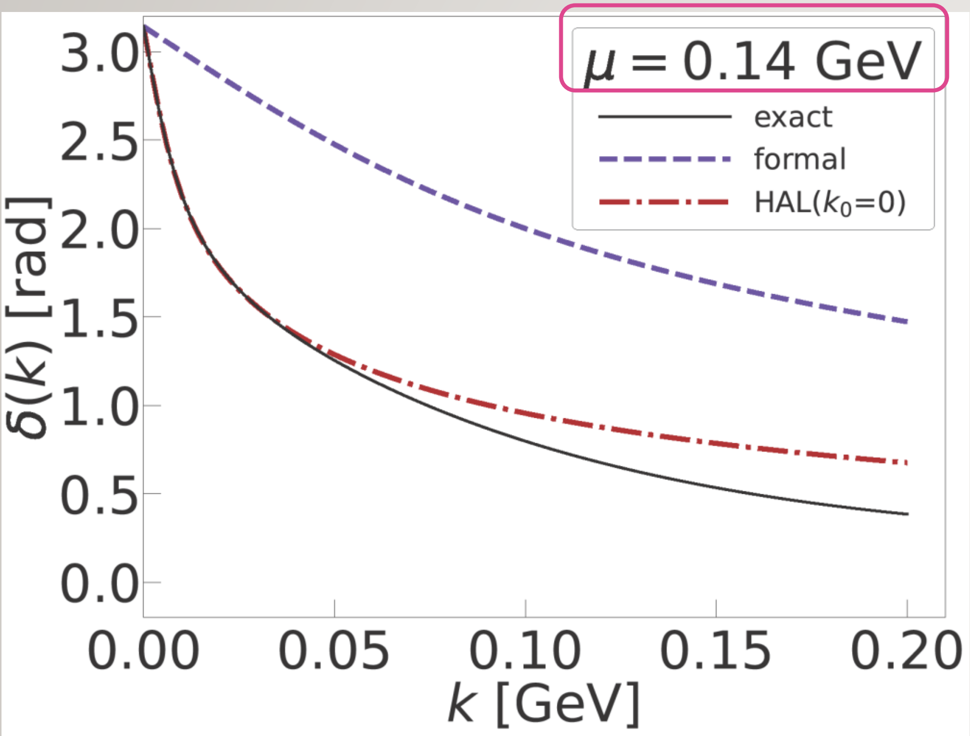
	formal	HAL QCD	exact
scattering length [fm]	6.55	24.48	24.48

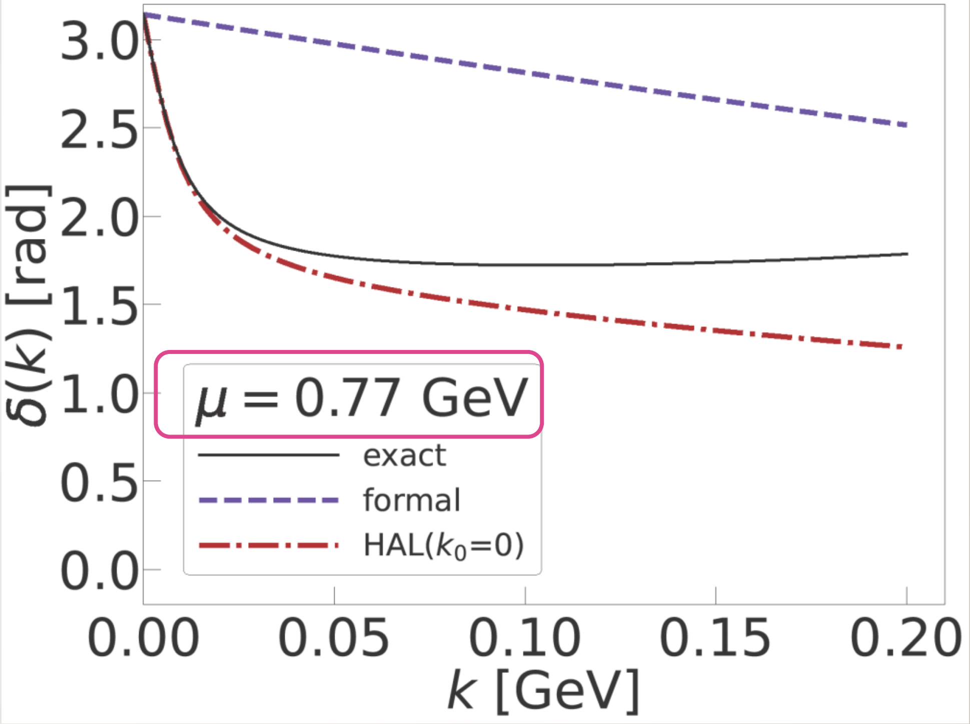


$\delta(k)$ as a function of dimensionless k/μ

Result : μ dependance of δ

- Change μ but fix the binding energy of $X(3872)$ and $k_0 = 0$





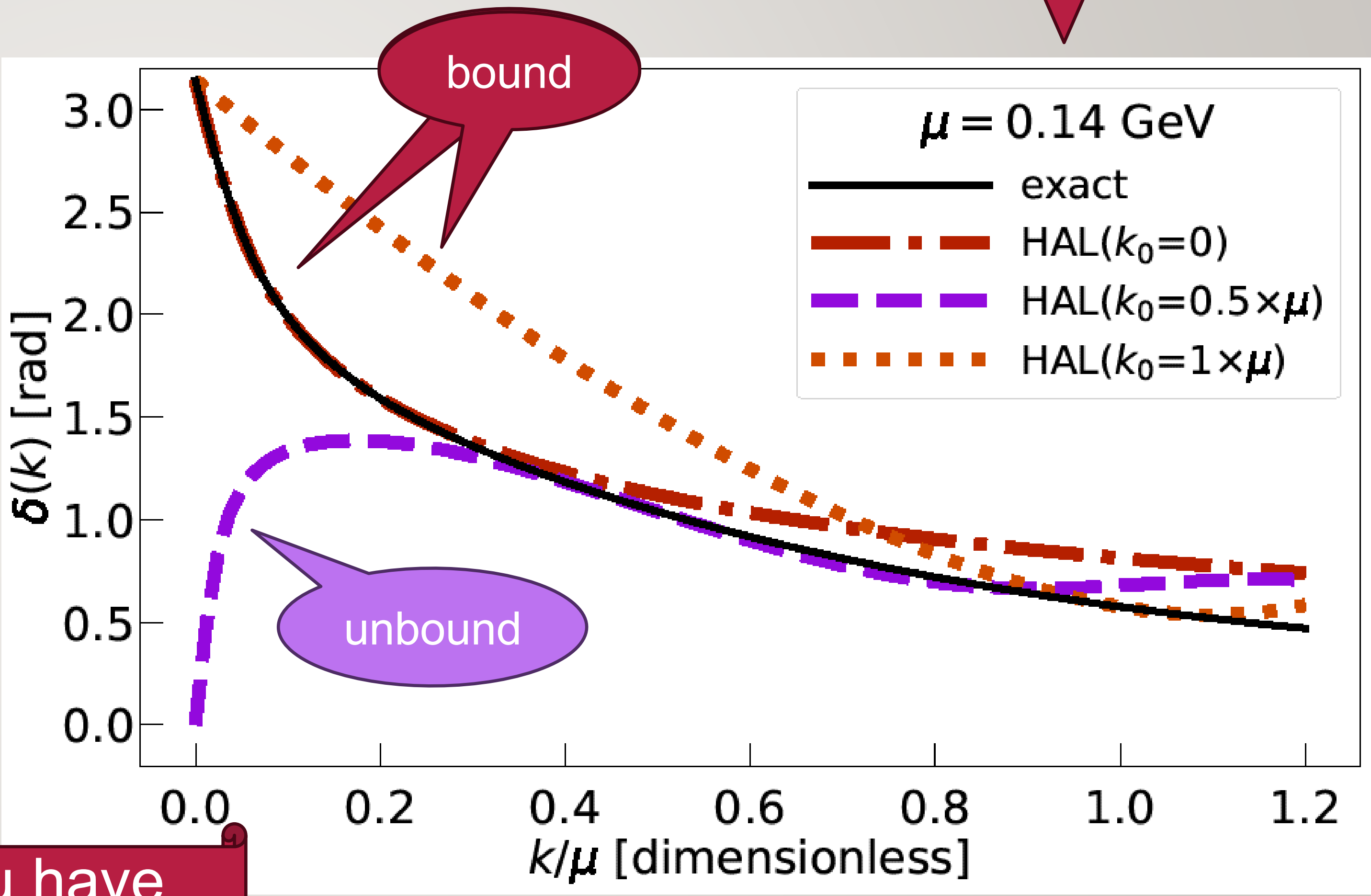
- At both μ, V^{HAL} reproduces exact $\delta(k)$ especially for low energy
- $V^{\mathrm{HAL}}(\boldsymbol{r}, E_{\mathrm{pot}} = 0)$ reproduce exact scattering length a_0 in any μ

Result : k_0 dependence of $\delta(k)$

Our Energy-dependent result $V_{\text{eff}}^{\bar{D}^* D}(\boldsymbol{r}, \boldsymbol{r}', E) \rightarrow V^{\text{HAL}}(r; k_0)$

- V^{HAL} does not so depend on k_0

- Phase shift $\boldsymbol{\delta}(\boldsymbol{k})$ from $V^{\text{HAL}}(r; k_0)$ qualitatively depends on k_0 strongly



Reason:

- Binding energy of $X(3872)$ is quite small (about 40 keV) so that the δ is sensitive to the choice of k_0

Take care when you have the shallow bound state

Summary

[I. Terashima and T. Hyodo, PhysRevC.108.035204 (2023)]

- Channel coupling between quark and hadron d.o.f → ◆Result : channel coupling
- Channel coupling between $c\bar{c}$ and $D\bar{D}^*$ in $X(3872)$
- Convert non-local E-dependent potential to local by (i) formal derivative expansion, (ii)HAL QCD method

➢ Energy dependent
➢ Non-local potential

✓ V^{formal} and V^{HAL} are quantitatively different

➢ $V^{\mathrm{HAL}}(r;k_0)$ reproduces the exact $\delta(k)$ better than V^{formal}

✓ Changing μ does not affect above result

✓ $V^{\mathrm{HAL}}(r;k_0)$ has quite small k_0 dependence

➢ $\delta(k)$ from $V^{\mathrm{HAL}}(r;k_0)$ qualitatively depends on k_0 strongly

Future outlook ◆ Append hadron-hadron interaction so that the study will become more realistic

appendix

Local and non-local

- Classified by matrix element of potential operator V in coordinate representation

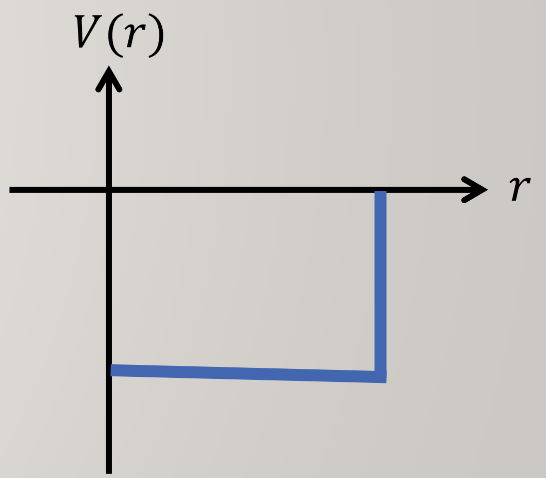
- **Non-local potential** $\langle \boldsymbol{r}' \mid V \mid \boldsymbol{r} \rangle = V(\boldsymbol{r}', \boldsymbol{r})$

$$-\frac{1}{2m}\nabla^2\psi(\boldsymbol{r}) + \int d^3\boldsymbol{r}'\; V(\boldsymbol{r}', \boldsymbol{r})\psi(\boldsymbol{r}') = E\psi(\boldsymbol{r})$$

➢More general potential, but the physics is not so clear

- **Local potential** $\langle \boldsymbol{r}' \mid V \mid \boldsymbol{r} \rangle = V(\boldsymbol{r})\delta(\boldsymbol{r}' - \boldsymbol{r})$

$$-\frac{1}{2m}\nabla^2\psi(\boldsymbol{r}) + V(\boldsymbol{r})\psi(\boldsymbol{r}) = E\psi(\boldsymbol{r})$$



➢Potential appeared in elementary quantum mechanics
Physical properties are well known (e.g., box potential)

Local approximations with Yukawa

- Formal derivative expansion at leading order

$$V^{\text{formal}}(r,E) = \omega(E)\frac{4\pi}{\mu^2}\frac{e^{-\mu r}}{r}.$$

- HAL QCD method at leading order

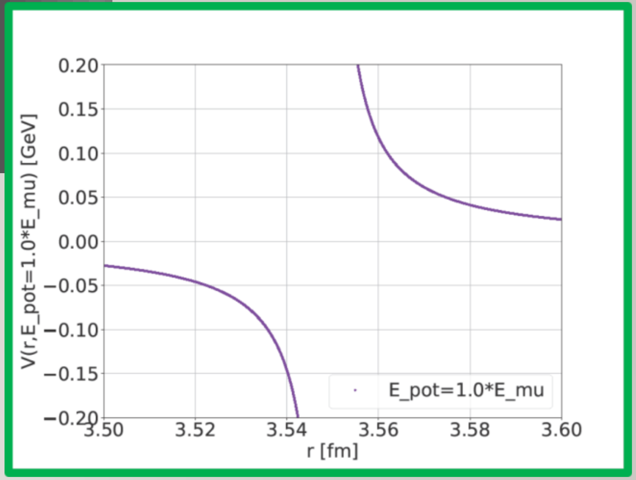
$$V^{\text{HAL}}(r;k_0) = \frac{k_0^2}{2m} + \frac{-k_0^2 \sin\left[k_0 r + \delta(k_0)\right] - \mu^2 \sin\delta(k_0) e^{-\mu r}}{2m\{\sin\left[k_0 r + \delta(k_0)\right] - \sin\delta(k_0) e^{-\mu r}\}}$$

- HAL QCD method at leading order in the limit of $k_0 \to 0$

$$V^{\text{HAL}}(r;k_0=0) = \frac{a_0 \mu^2 e^{-\mu r}}{2m\left(r - a_0 + a_0 e^{-\mu r}\right)}$$

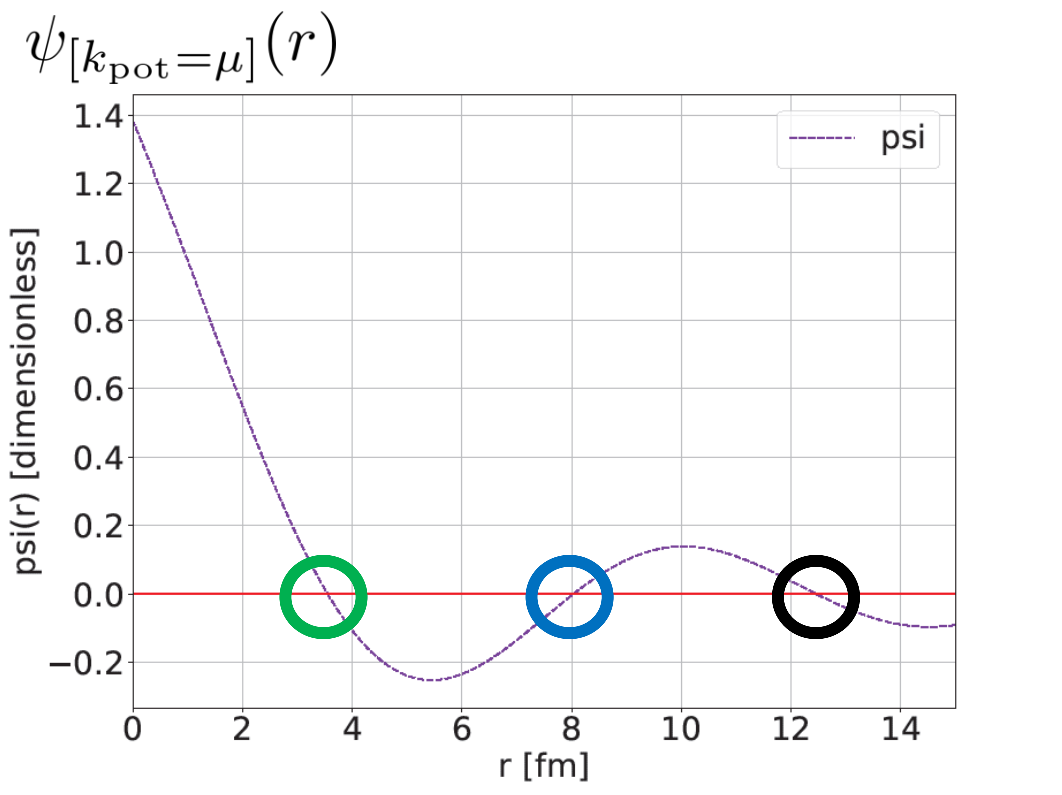
  - ➢Written only by a_0

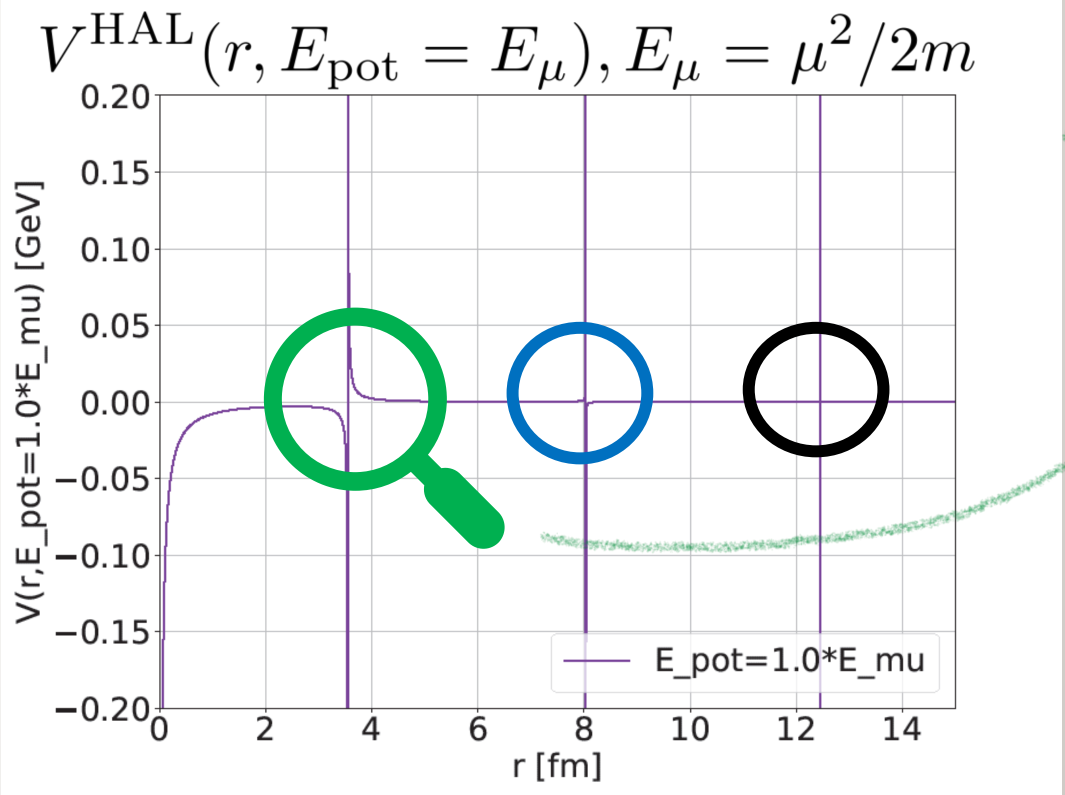
Note : HAL QCD method



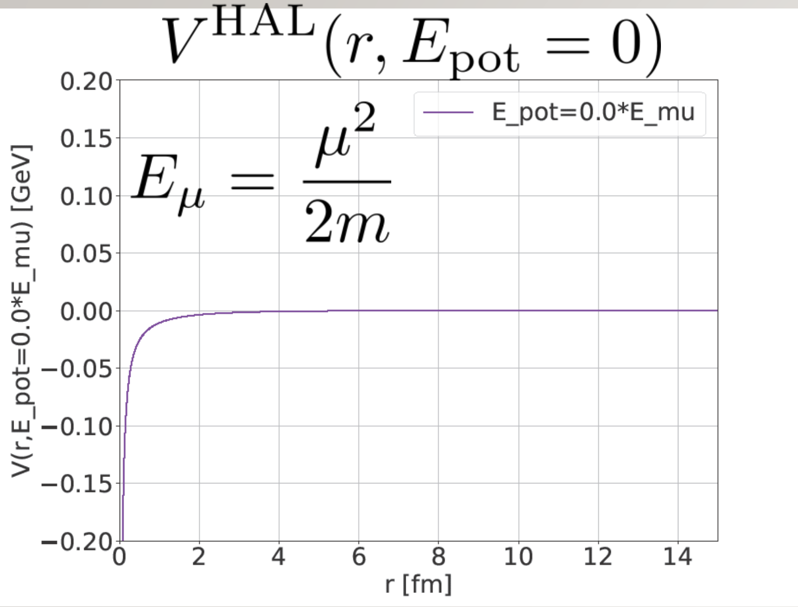
$$V^{\mathrm{HAL}}(r, E_{\mathrm{pot}}) = E_{\mathrm{pot}} + \frac{1}{2mr\psi_{k_{\mathrm{pot}}}(r)} \frac{d^2}{dr^2}\left[r\psi_{k_{\mathrm{pot}}}(r)\right] + \mathcal{O}(\nabla^2),$$

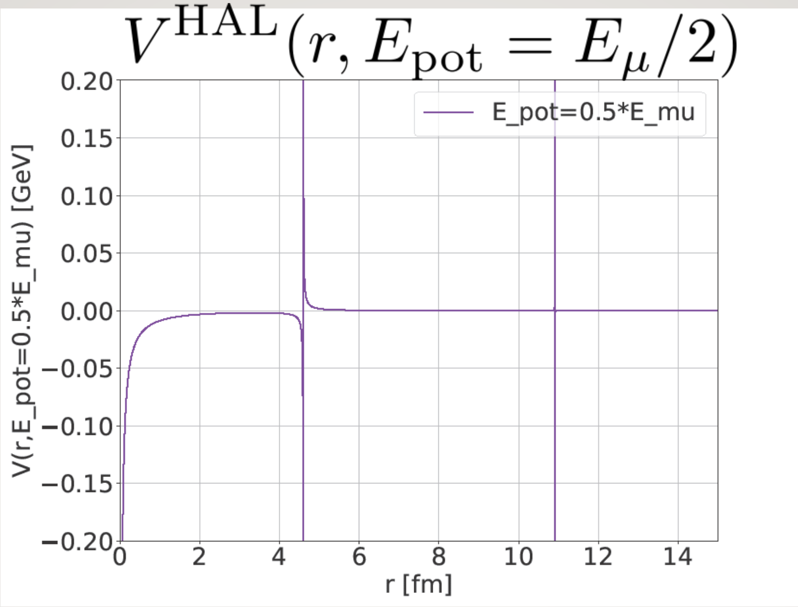
- Potential diverges at $\psi_{k_{\mathrm{pot}}}(r) = 0$

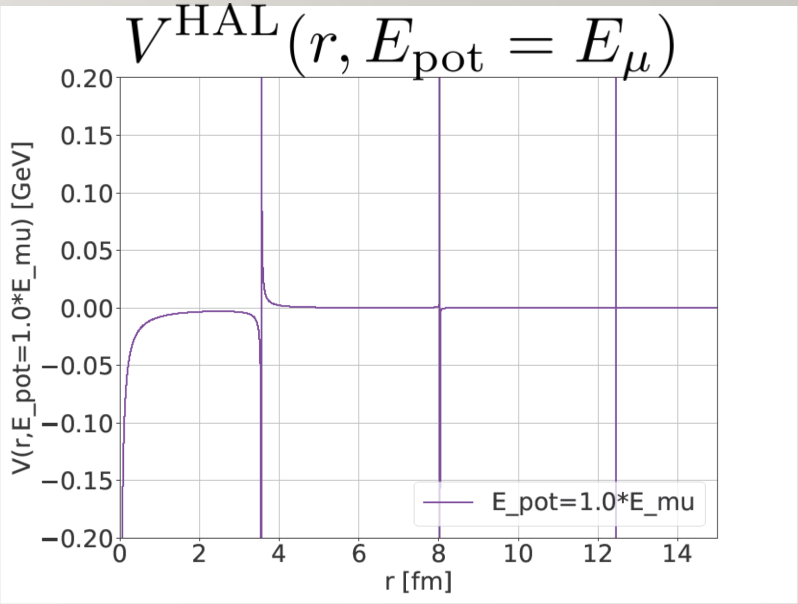




Result : energy dependance of divergence







- Change E_{pot} to investigate energy dependance of divergence of $V^{\mathrm{HAL}}(r, E_{\mathrm{pot}})$

➡ $V^{\mathrm{HAL}}(r, E_{\mathrm{pot}})$ diverges at smaller r when E_{pot} is larger

✓Because frequency $\frac{\sqrt{2mE_{\mathrm{pot}}}}{\hbar}$ is getting higher when E is larger

Result : E dependance of V^{formal} for $X(3872)$

- Local potentials
  by formal derivative expansion

$$V_{\rm eff}^{\bar{D}^*D}(\boldsymbol{r},\boldsymbol{r}',E) \to V^{\rm formal}(r,E)$$

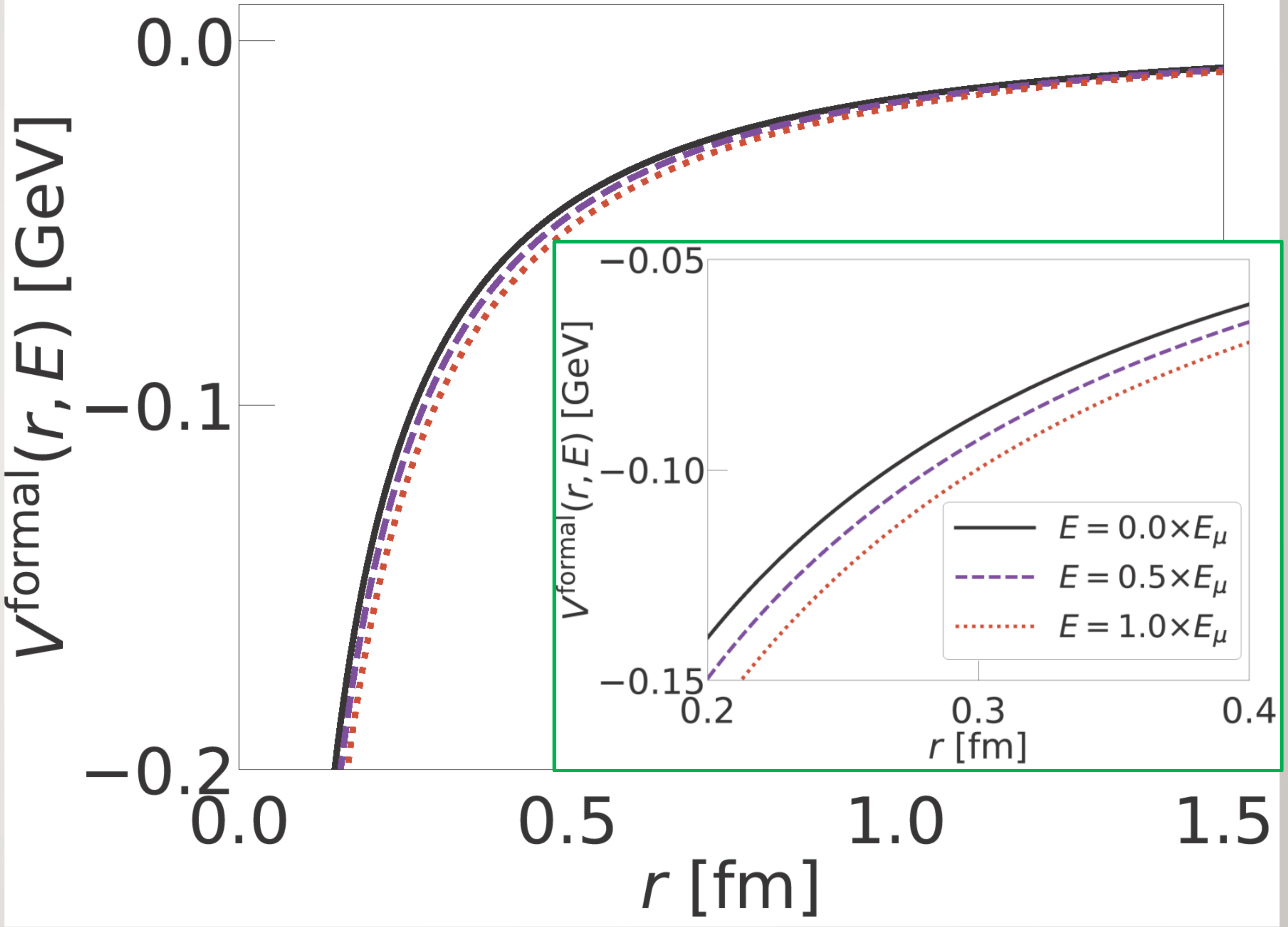
Small E dependance

Reason:

$$V^{\rm formal}(r,E) = \frac{4\pi g_0^2}{\mu^2(E-E_0)}\frac{e^{-\mu r}}{r}$$

$$E \ll E_0 = 0.078\text{GeV}$$

$(E_\mu \simeq 0.01\ \text{GeV})$



μ : cut-off (0.14GeV) $E_\mu = \mu^2/2m$

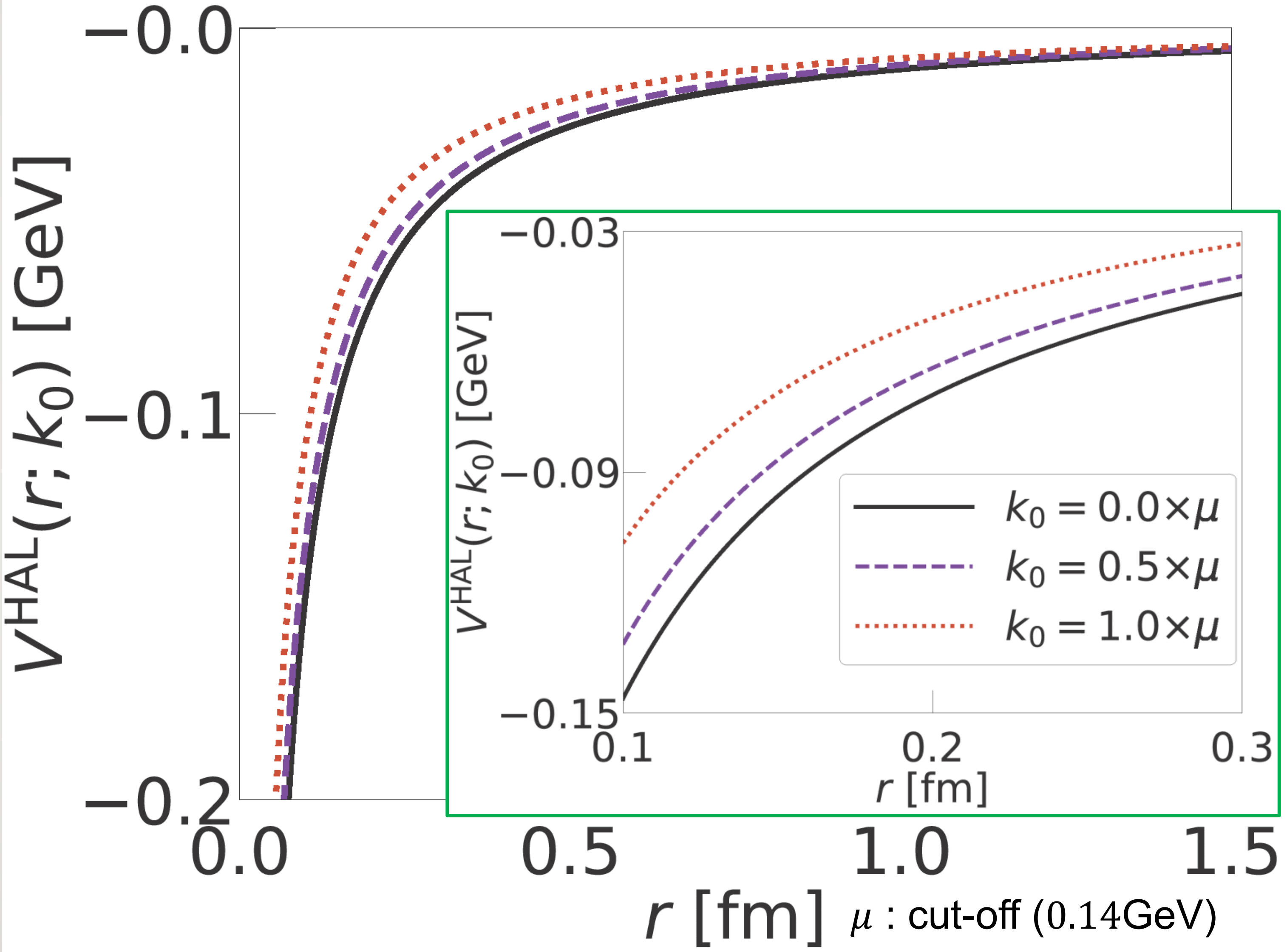
Result : k_0 dependance of V^{HAL} for $X(3872)$

- Local potentials
  by HAL QCD method

$$V_{\text{eff}}^{\bar{D}^* D}(\boldsymbol{r}, \boldsymbol{r}', E) \rightarrow V^{\text{HAL}}(r; k_0)$$

Qualitatively small

k_0 dependance



μ : cut-off (0.14GeV)

k_0 dependence of a_0

■Scattering lengths a_0 are calculated at $k = 0$

■Reproduce the exact $\delta(k)$ at $k = k_0$

- ➢NOT guaranteed to reproduce at $k \neq k_0$
- ➢$\delta(k)$ around $k = 0$ is NOT fixed

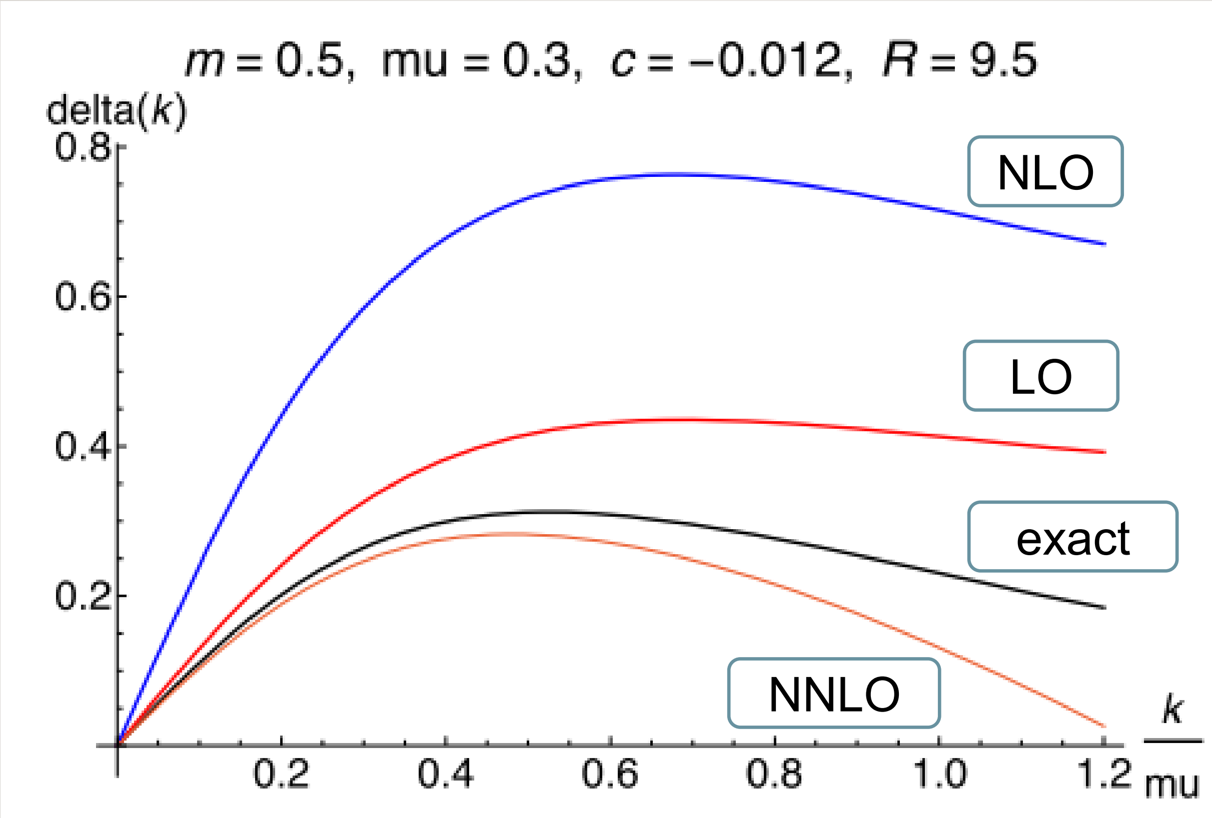
■Strong k_0 dependence of a_0

TABLE II. The k_0 dependence of the scattering length a_0 from the potential by the HAL QCD method, with $\mu = m_\pi = 0.14$ GeV and $\mu = m_\rho = 0.77$ GeV.

k_0/μ [dimensionless]	$a_0(\mu = m_\pi)$ [fm]	$a_0(\mu = m_\rho)$ [fm]
0	24.48	22.36
0.1	24.14	8.32
0.2	21.38	2.84
0.3	22.68	1.34
0.4	17.17	0.79
0.5	−63.97	0.71
0.6	9.33	0.01
0.7	5.88	0.23
0.8	−0.78	0.60
0.9	−1.27	−0.13
1	5.21	−0.20

Formal derivative expansion

- The previous study pointed out that the formal derivative expansion does not reproduce exact phase shift $\delta(k)$ at NLO so that they introduced the HAL QCD method



NOT improving order by order

N^3LO and N^4LO also do not improve approximation at all

[S.Aoki and K.Yazaki, PTEP 2022, no.3, 033B04 (2022)]

Result : effective potential

- Coordinate representation with initial relative coordinate $\boldsymbol{r}$ and final $\boldsymbol{r}'$

$$\langle \boldsymbol{r}'_h \mid V^h_{\text{eff}}(E) \mid \boldsymbol{r}_h \rangle = \langle \boldsymbol{r}'_h \mid V^h \mid \boldsymbol{r}_h \rangle + \sum_n \frac{\langle \boldsymbol{r}'_h \mid V^t \mid \phi_n \rangle \langle \phi_n \mid V^t \mid \boldsymbol{r}_h \rangle}{E - E_n}$$

➢ Quark channel contribution. Sum of discrete eigenstates E_n

$$\langle \boldsymbol{r}'_q \mid V^q_{\text{eff}}(E) \mid \boldsymbol{r}_q \rangle = \langle \boldsymbol{r}'_q \mid V^q \mid \boldsymbol{r}_q \rangle + \int d\boldsymbol{p} \frac{\langle \boldsymbol{r}'_q \mid V^t \mid \boldsymbol{p}_{\text{full}} \rangle \langle \boldsymbol{p}_{\text{full}} \mid V^t \mid \boldsymbol{r}_q \rangle}{E - E_{\boldsymbol{p}} + i0^+}$$

➢ Hadron channel contribution. Integral of continuous eigenstates E_p

◆ Energy dependent potential (denominator depends on E)
◆ Non-local potential (numerator depends on $\boldsymbol{r}, \boldsymbol{r}'$ independently)

✓Focus only on the 2nd term which represents the contribution of channel coupling

Result

- Consider an imaginary part of the effective potential of quark $V^q_{\text{eff}}(E)$

$$\langle \boldsymbol{r}'_q \mid V^q_{\text{eff}}(E) \mid \boldsymbol{r}_q \rangle = \underline{\langle \boldsymbol{r}'_q \mid V^q \mid \boldsymbol{r}_q \rangle} + \int d\boldsymbol{p} \frac{\langle \boldsymbol{r}'_q \mid V^t \mid \boldsymbol{p}_{\text{full}} \rangle \langle \boldsymbol{p}_{\text{full}} \mid V^t \mid \boldsymbol{r}_q \rangle}{E - E_{\boldsymbol{p}} + i0^+}$$

$\in \mathbb{R}$

Different from $V^h_{\text{eff}}(E)$, $V^q_{\text{eff}}(E)$ induces an imaginary part when $E \geq \Delta$

$$\text{Im}\left[\langle \boldsymbol{r}'_q \mid V^q_{\text{eff}}(E) \mid \boldsymbol{r}_q \rangle\right] = 4\pi^2 m \sqrt{2m(E-\Delta)} \times \langle \boldsymbol{r}'_q \mid V^t \mid \boldsymbol{p}_{\text{full}} \rangle \langle \boldsymbol{p}_{\text{full}} \mid V^t \mid \boldsymbol{r}_q \rangle \Theta(E-\Delta)$$

This is because the integrant has a pole at $E = E_{\boldsymbol{p}}$.

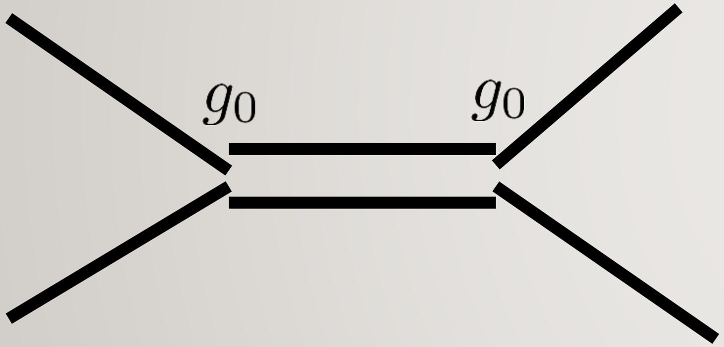
- When the potential has an imaginary part, the V is not Hermitian
- This represents the decay process into a hadron channel properly

⇔ Qualitatively different from the static limit (string-breaking)

Preliminary

$$\omega(E) = \omega^h + \omega^q = \omega^h + \frac{g_0^2}{E - E_0}$$

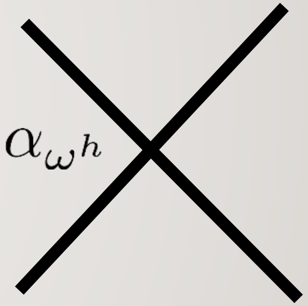
$$\omega^q = \frac{g_0^2}{E - E_0}$$



elementary

+

$$\omega^h = \alpha_{\omega^h} \times \frac{g_0^{q\,2}}{-E_0}$$



molecule

ハドロン間相互作用を加える

- 分離型のハドロン間相互作用を加える

$$H = \begin{pmatrix} T^q & 0 \\ 0 & T^h + \Delta \end{pmatrix} + \begin{pmatrix} V^q & V^t \\ V^t & V^h \end{pmatrix}$$

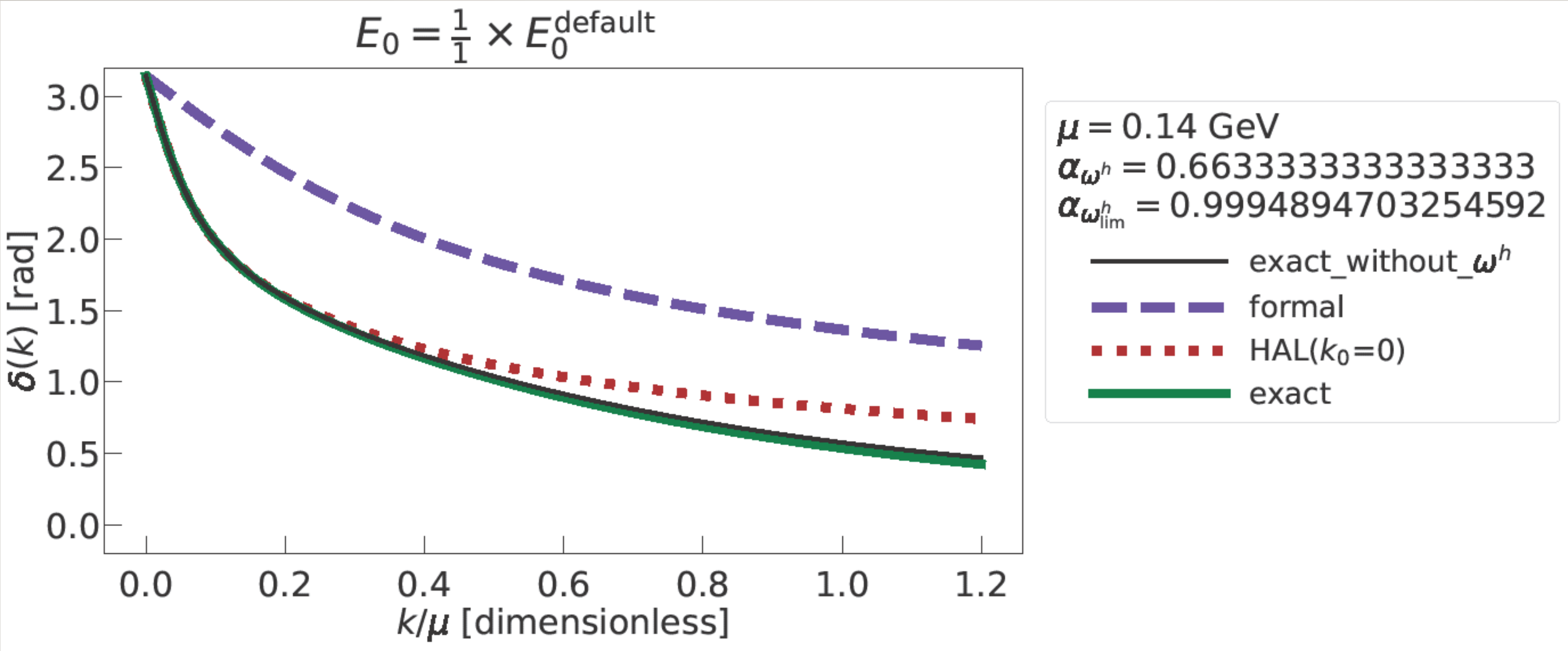
$$V^h = \omega^h V(\boldsymbol{r}')V(\boldsymbol{r}) \qquad \omega^h = \alpha_{\omega^h} \times \frac{g_0^{q\,2}}{-E_0}$$

- このとき、フルなポテンシャル強度ωは、

$$\omega(E) = \omega^h + \omega^q = \omega^h + \frac{g_0^2}{E - E_0} \qquad g_0^{q\,2} = g_0^2(\omega^h = 0)$$

$$V_{\text{eff}}^{\bar{D}^* D}(\boldsymbol{r}, \boldsymbol{r}', E) = \omega(E)V(\boldsymbol{r})V(\boldsymbol{r}')$$

結果



- (特に低エネルギーで)HAL QCD法の方が、formal微分展開よりも厳密な位相差を再現する
- V^hを加えていないときと同じ結果

Result : Divergence of δ

$$V^{\mathrm{formal}}(r,E) = \frac{4\pi g_0^2}{\mu^2(E-E_0)}\frac{e^{-\mu r}}{r}$$

- δ from V^{formal} diverge at $E = E_0$
- δ from V^{HAL} doesn't diverge

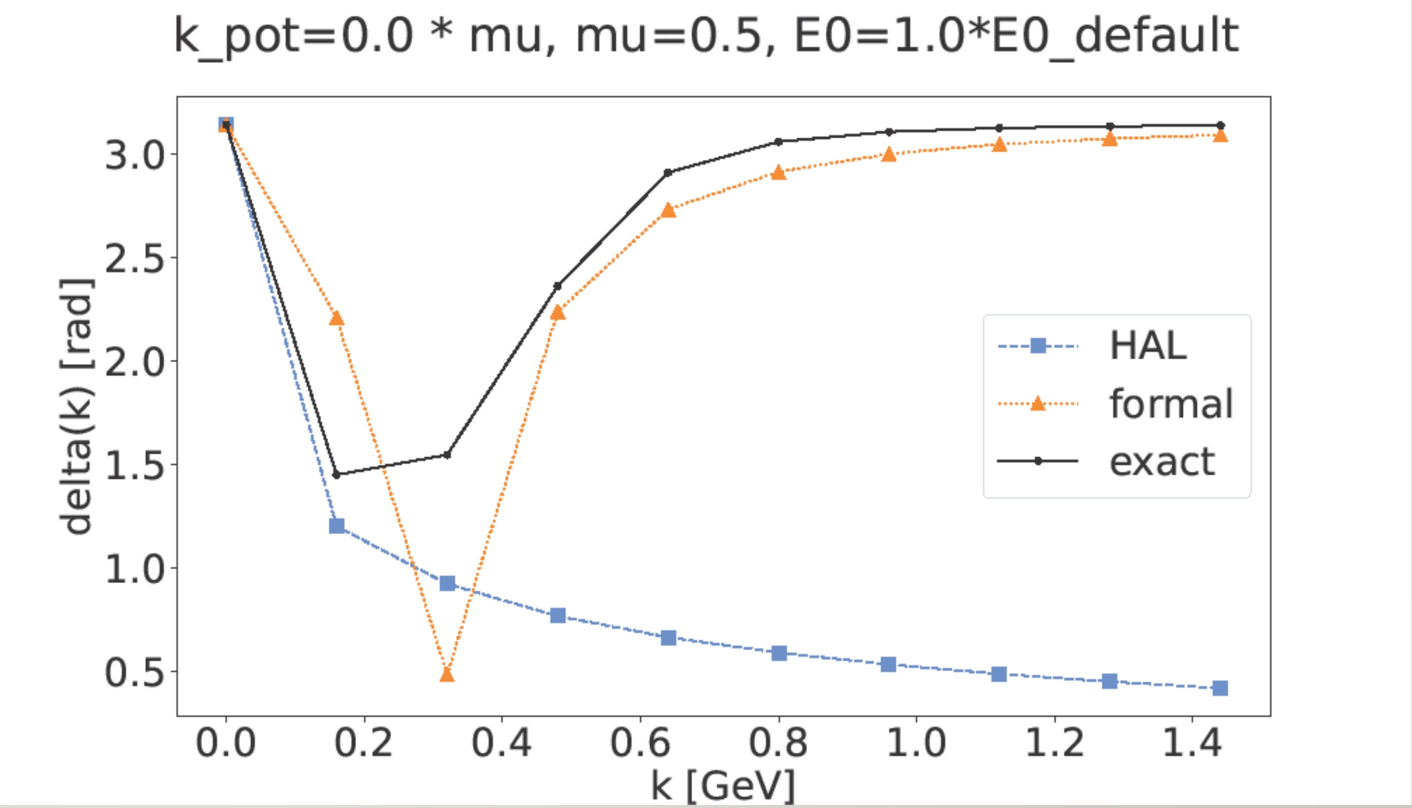


表 6.1　パラメーターの値

物理量	値	出典
$m_{c\bar{c}}$	3.950 GeV	クォーク模型の値 [24]
m_{D^0}	1.86484 GeV	PDG [60]
$m_{D^{0*}}$	2.00685 GeV	PDG [60]
μ	0.14 GeV	PDG [60]
$\hbar c$	0.1973269804 GeV · fm	PDG [60]
$m_{X(3872)}$	3.87165 GeV	PDG [60]

$\chi_{c1}(2P)$

表 6.2　数値計算で得られたパラメーター

物理量	値	計算式
E_0	0.07831 GeV	$m_{c\bar{c}} - m_{D^0} - m_{\bar{D}^{*0}}$
m	0.9666 GeV	$\frac{m_{D0}+m_{\bar{D}*0}}{m_{D0}m_{\bar{D}*0}}$
g_0^2	1.999×10^{-5} GeV^3	$\frac{m_{c\bar{c}}-m_{X(3872)}}{I}$
a_0	124.1 fm	$\frac{8\pi m g_0^2/E_0}{\mu(4\pi m g_0^2/E_0+\mu^3)}$