Mittag-Leffler Expansion to Hadron Physics

ELPH Workshop C033, December 7, 2022

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W. Yamada, O. Morimatsu, Phys. Rev. C 102, 055201 (2020)
W. Yamada, O. Morimatsu, Phys. Rev. C 103, 045201 (2021)
W. Yamada, O. Morimatsu, T. Sato, K. Yazaki, Phys. Rev. D 105, 014034 (2022)
W. Yamada, O. Morimatsu, T. Sato, Phys. Rev. Lett. 129, 192001 (2022)

Contents

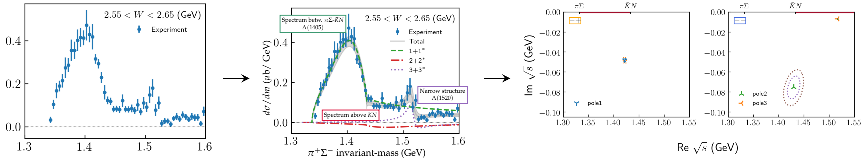
- Analytic structure of S-matrix and shape of poles:

Energy Region	Structure of S-matrix	Shape
Distant from threshold	Trivial ('flat') in Energy	Breit-Wigner ○
Near threshold	Non-trivial in Energy	Breit-Wigner ×

Objective:

Clarify analytic stucture of 2, 3-channel S-matrix: draw a 'map' of the S-matrix

Near-threshold spectrum decomposition & extraction of pole properties



- Uniformization: mapping the non-trivial analytic structure of S-matrix
- Mittag-Leffler Expansion: pole expansion of meromorphic functions

Uniformization

<u>Uniformization</u>

'Multi-sheeted' Riemann surface (e.g. Energy-parameterization)

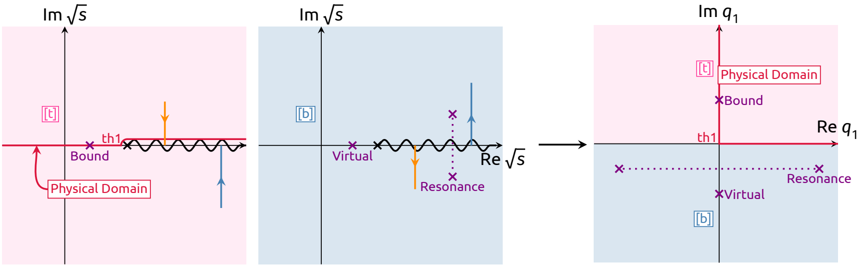
↓ Conformal Map

Uniformized Plane

Trivial surface (as much as possible), entire or partial region of a complex plane

- Analytic structure preserved
- Globally 'flat' nature (constant curvature) → clarifies "distance"

■ Example: Single-channel S-matrix 2-body, RH cuts and poles



Mittag-Leffler Expansion

Mittag-Leffler Expansion: Pole expansion of meromorphic functions

Corollary from Mittag-Leffler Theorem,

$F(z)$: meromorphic, $\sum_n |r_n|/|z_n|^{m+1}$: finite z_n, r_n : pole position, residue

$$F(z) = z^m \sum_n \frac{z_n^{-m} r_n}{z - z_n} + E(z)$$

Pole terms   Subtraction terms

- Example: $\cot z = \sum_{n=-\infty}^{\infty} \frac{1}{z - n\pi}$

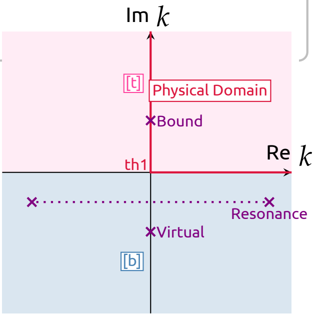
Single-channel S-matrix

Mittag-Leffler Expansion by momentum k

J. Humblet, L.Rosenfeld, Nucl. Physics 26 (1961)

D. Ramírez Jiménez, N. Kelkar, Annals of Physics 396, 18 (2018)

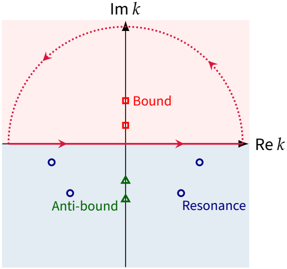
$$A(k) = k^m \sum_n \left[\frac{k_n^{-m} r_n}{k - k_n} + \frac{(-k_n^*)^{-m} r_n^*}{k + k_n^*} \right] + Q(k)$$



Resonant-state Expansion of the Resolvent

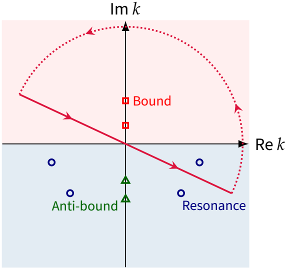
T. Berggren, P. Lind, Phys. Rev. C 47, 768 (1993)

- **Spectral Expansion**



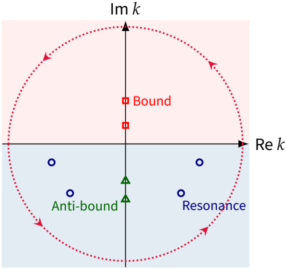
$$\hat{\mathcal{G}}(k) = \underbrace{\sum_n \frac{|\kappa_n\rangle\langle\kappa_n|}{k - i\kappa_n}}_{\text{Bound}} + \underbrace{\int_C dk_c \frac{|k_c\rangle\langle k_c|}{k - k_c}}_{\text{Continuum}}$$

- **Berggren Expansion**



$$\hat{\mathcal{G}}(k) = \underbrace{\sum_n \frac{|\kappa_n\rangle\langle\kappa_n|}{k - i\kappa_n}}_{\text{Bound}} + \underbrace{\sum_m{}' \frac{|k_m\rangle\langle\tilde{k}_m|}{k - k_m}}_{\text{Resonance}} + \underbrace{\int_{C'} dk_c' \frac{|k_c'\rangle\langle\tilde{k}_c'|}{k - k_c'}}_{\text{modified Continuum}}$$

- **Mittag-Leffler Expansion**

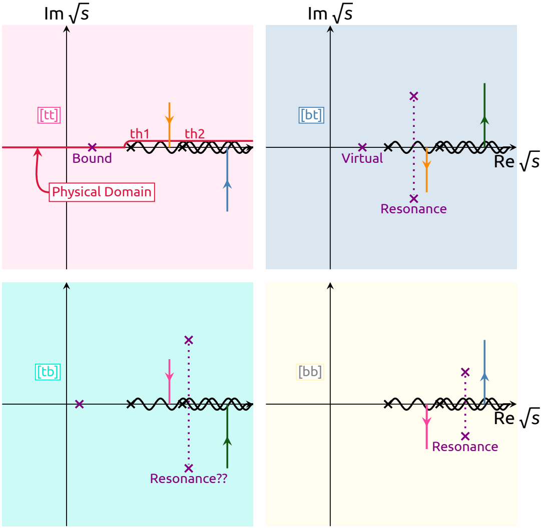


$$\hat{\mathcal{G}}(k) = \underbrace{\sum_n \frac{|\kappa_n\rangle\langle\kappa_n|}{k - i\kappa_n}}_{\text{Bound}} + \underbrace{\sum_l \frac{|\kappa_l'\rangle\langle\kappa_l'|}{k + i\kappa_l'}}_{\text{Anti-bound}} + \underbrace{\sum_m \frac{|k_m\rangle\langle\tilde{k}_m|}{k - k_m}}_{\text{Resonance}}$$

Riemann Sphere representation of the 2-channel S-matrix

- RS of 2-channel S-matrix 2-body, RH cuts and poles
  - 4-sheeted $\sqrt{s}$-plane: [tt], [bt], [tb], [bb]

e.g. $[tb]_+$ means $Im\, q_1 > 0$, $Im\, q_2 > 0$, and $Im\sqrt{s} > 0$



Riemann Sphere representation of the 2-channel S-matrix

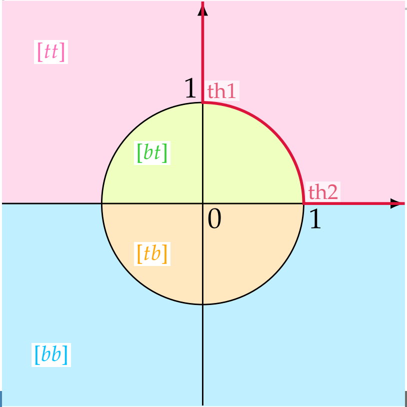
z plane:

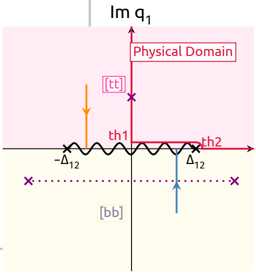
M. Kato, Ann. Phys. 31, 130 (1965)

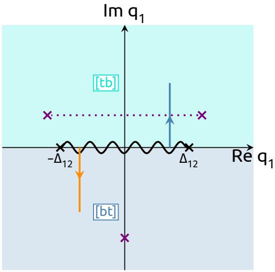
$$z = \frac{1}{\Delta_{12}}(q_1 + q_2),$$

$$q_i = \sqrt{s - \epsilon_i^2},\ \Delta_{12} = \sqrt{\epsilon_2^2 - \epsilon_1^2}$$

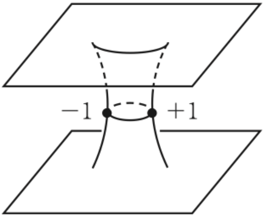
ϵ_1, ϵ_2 channel mass

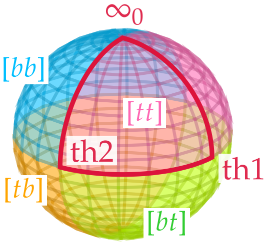




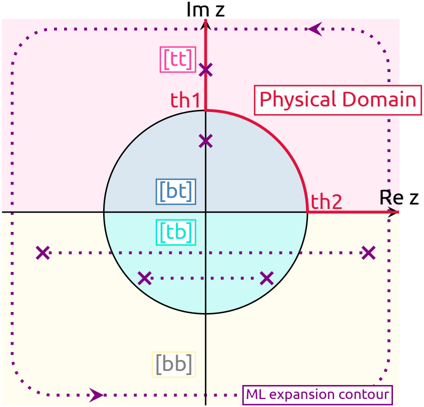


Stereographic Projection of Sphere from 'North Pole'(∞_0) → Equatorial Plane





Mittag-Leffler Expansion on the Riemann Sphere



2-channel Mittag-Leffler Expansion

$$A(z) = \sum_i \frac{r_i}{z - z_i} + \text{(subtraction)}$$

W. Y., O.M., PRC 102, 055201 (2020)

- Unitarity of the S-matrix:

$$S(-z^*) = S(z)^*$$

Symmetric poles about Im z-axis
→ appropriate threshold behavior

- Lone pole-pair contribution:

$$A_n = \frac{r_n}{z - z_n} - \frac{r_n^*}{z + z_n^*}$$

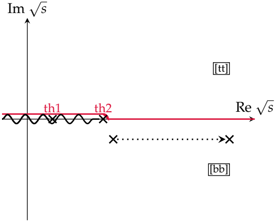
$$\operatorname{Im} A_n(z) = \begin{cases} 0, & (\sqrt{s} < \epsilon_1) \\ -\operatorname{Im} \frac{2r_n}{(z_n - i)^2} \boxed{\frac{q_1}{\Delta}} + O(q_1^2), & (\sqrt{s} > \epsilon_1) \end{cases}$$

$$\operatorname{Im} A_n(z) = \begin{cases} \operatorname{Im} \frac{2r_n}{1 - z_n^2} - \operatorname{Re} \frac{4r_n z_n}{(1 - z_n^2)^2} \boxed{\frac{\tilde{q}_2}{\Delta}} + O(\tilde{q}_2^2), & (\sqrt{s} < \epsilon_2) \\ \operatorname{Im} \frac{2r_n}{1 - z_n^2} - \operatorname{Im} \frac{2r_n(1 + z_n^2)}{(1 - z_n^2)^2} \boxed{\frac{q_2}{\Delta}} + O(q_2^2), & (\sqrt{s} > \epsilon_2) \end{cases}$$

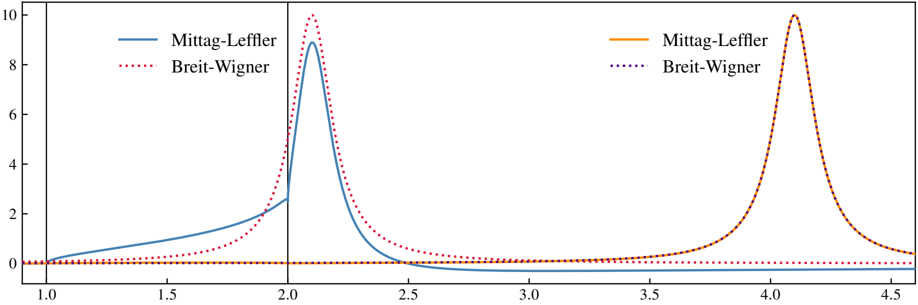
- Lone pole-pair contribution, A_n

Under the condition, $|z_R| >> 1, \zeta_R >> \eta_R$, where $z_R = \zeta_R + i\eta_R$,

$$\left.\frac{r_R}{z - z_R} - \frac{r_R^*}{z + z_R^*}\right|_{z \simeq \zeta_R} \simeq \frac{\Delta\left[r_R \zeta_R + (r_R - r_R^*) i\eta_R/2\right]}{4(2m_R + i\Gamma_R/2)} \frac{1}{\sqrt{s} - m_R + i\Gamma_R/2}$$



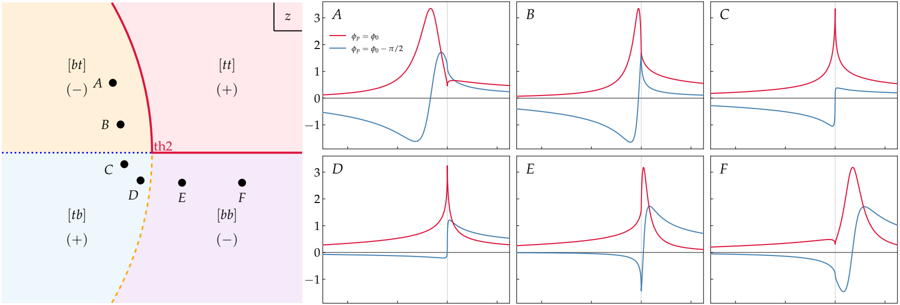
$\epsilon_1 = 1.0, \epsilon_2 = 2.0, \Gamma_R = 0.2, m_R = 2.1,\ 4.1$



- Approriate threshold behaviors
- Coincides with Breit-Wigner Form in the distant-to-threshold limit

Lineshapes: Pole at upper threshold

Wren A. Yamada, Osamu Morimatsu, Toru Sato, Koichi Yazaki Phys. Rev. D 105, 014034 (2022)



	A	B	C	D	E	F
z_p	$0.869 + 0.233i$	$0.895 + 0.094i$	$0.908 - 0.038i$	$0.962 - 0.092i$	$1.100 - 0.100i$	$1.300 - 0.100i$
e_p	$0.943 - 0.053i$	$1.000 - 0.022i$	$1.007 + 0.008i$	$0.992 + 0.007i$	$1.002 - 0.018i$	$1.065 - 0.043i$
e_0	0.933	0.989	1	1	1.01	1.065
γ_0	0.100	0.100	0.100	0.100	0.100	0.100
Sheet	$[bt]$	$[bt]$	$[tb]$	$[tb]$	$[bb]$	$[bb]$

- Resonance: $[bt]_-$, $[bb]_-$sheet, "Threshold Cusp": $[tb]_+$sheet
- 'Peak position', 'width' = closest physical point, and its distance <u>on *z*</u> ≠ Re *E*, Im *E*

Flatté Formula on the Riemann Sphere

Flatté Formula

S. M. Flatté, Phys. Lett., B63, 224 (1976)

$$A_{11} = \frac{-\gamma_1 k_1}{E - m + i\gamma_1 k_1 + i\gamma_2 k_2}$$

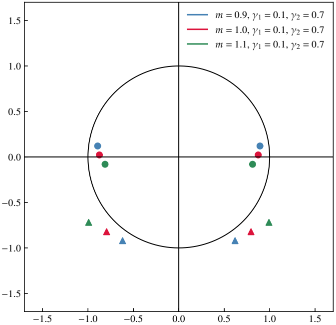
$$A_{11} = -\gamma \frac{z^3 + z}{z^4 + 1 + \alpha z^2 + i[(\gamma + \gamma')z^3 + (\gamma - \gamma')z]}$$

$$\gamma = 4\frac{\gamma_1 \mu_1}{\Delta},\ \gamma' = 4\frac{\gamma_2 \sqrt{\mu_1 \mu_2}}{\Delta},\ \alpha = 4\frac{\mu_1}{\Delta^2}[\varepsilon_1 + \varepsilon_2 - 2m]$$

$$A_{11} = \sum_{i=1,2}\left[\frac{r_i}{z - z_i} + \frac{r_i^*}{z + z_i^*}\right]$$

2-pole Mittag-Leffler Expansion

pole position condition: $|z_1 z_2| = 1$



- Always contain 2 pair of poles: one on [tb]/[bt], the other on [bb]-sheet
- At inelastic threshold: complex scattering length, complex effective range → 4 parameters (Flatté has 3. Leading to additional constraint $|z_1 z_2| = 1$)

Uniqueness of MLE Pole Decomposition

- Mittag-Leffler Expansion: Pole decomposition on the uniformized plane
- 2-channel uniformization plane: non-unique

  Smooth bijective mapping $CP^1 \to CP^1$ induces new uniformization plane

$$Aut(\mathbb{CP}^1) \cong PGL(2, \mathbb{C})$$

$$z \mapsto w = \frac{\alpha z + \beta}{\gamma z + \delta}, \text{ where, } det \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} \neq 0$$

- Question: Is the pole decomposition on *z*-plane and *w*-plane identical?

Ex: $\zeta = 1/z$ z (ζ): "north pole" ("south pole") projection of Riemann Sphere

$$\frac{r_n^{[z]}}{z - z_n} = -r_n^{[z]} \frac{\zeta \zeta_n}{\zeta - \zeta_n} = \frac{r_n^{[\zeta]}}{\zeta_n} \frac{\zeta}{\zeta - \zeta_n} = \frac{r_n^{[\zeta]}}{\zeta - \zeta_n} + \frac{r_n^{[\zeta]}}{\zeta_n}$$

Pole term on z Pole term on ζ

Pole term has indefiniteness of a constant fixed by imposing boundary condition: $A_n \to 0$ as approaching the infinity point on physical sheet

- MLE Pole decomposition unique under $CP^1 \to CP^1$ transformations

Uniqueness of MLE Pole Decomposition

Non-algebraic mapping $CP^1 \to C/Z$: "Cylinder" representation

$$\eta \mapsto \omega = \int_0^\eta \frac{d\zeta}{1+\zeta^2} = \arctan \eta \ (\eta \neq \pm i)$$

- Periodicity: $A(\omega + \pi\mathbb{Z}) = A(\omega)$
- Mittag-Leffler Expansion:

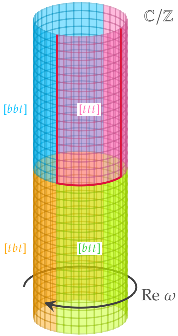
$$A(\omega) = \sum_n r_n \left[\frac{1}{\omega - \omega_n} + \sum_{m \neq 0} \frac{1}{\omega - \omega_n + m\pi} \right] = \sum_n r_n \cot(\omega - \omega_n)$$

Corrections

Pole term

$$\frac{r_p^{[\eta]}}{\eta - \eta_p} = \frac{(1+\tan^2 \omega_p)\, r_p^{[\omega]}}{\tan\omega - \tan\omega_p}$$
$$= r_p^{[\omega]} \frac{1 + \tan\omega \tan\omega_p - \tan\omega_p(\tan\omega - \tan\omega_p)}{\tan\omega - \tan\omega_p}$$
$$= r_p^{[\omega]} \cot(\omega - \omega_p) - r_p^{[\omega]} \tan\omega_p.$$

$$Aut(\mathbb{C}/\mathbb{Z}) : z \mapsto \begin{cases} z + a \\ -z \end{cases}$$



- MLE pole decomposition is unique under many different uniformization planes
- Obtain same results when fitting observables by a truncated MLE regardless of the choise of the uniformization plane (At least for CP^1, C/Z)

Λ(1405)

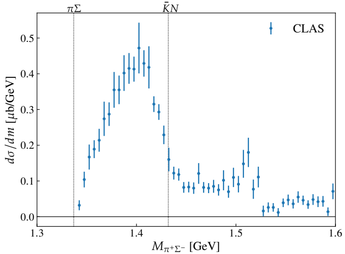
Wren Yamada, Osamu Morimatsu, PRC 103, 045201

- $\gamma p \to K^+ \pi\Sigma$

  K. Moriya, et al. CLAS, PRC 87, 035206 (2013)

- $K^- p \to K^- p, \bar{K}^0 n, \pi^\pm \Sigma^\mp$

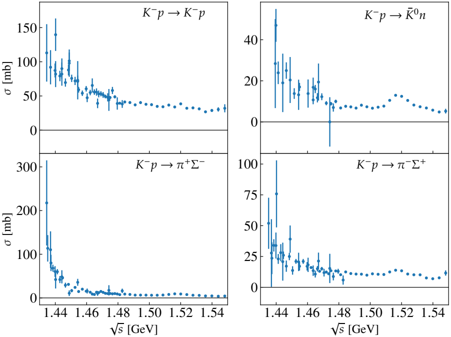
  Abrams et al. Phys. Rev. 139, B454 (1965),
  Bangerter et al. Phys. Rev. D 23, 1484 (1981),
  Ciborowski et al. J. Phys. G: Nucl. Phys. 8, 13 (1982),
  Csejthey-Barth et al. Phys.Lett. 16, 89 (1965),
  Humphrey et al. Phys. Rev. 127, 1305 (1962),
  Mast et al. Phys. Rev. D 14, 13 (1976),
  Sakitt et al. Phys. Rev. 139, B719 (1965)



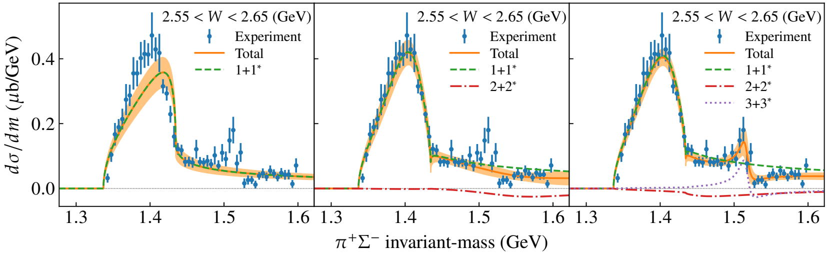
- Sphere of 2-channel system: $\pi\Sigma$, $\bar{K}N$
- 3-pole Mittag-Leffler Expansion, common poles

$$\frac{d\sigma^{(\pi\Sigma)}}{dm} = \mathrm{Im} \sum_{n=1}^{3} \left[\frac{c_n^{(\pi\Sigma)}}{z - z_n} - \frac{c_n^{(\pi\Sigma)*}}{z + z_n^*} \right],$$

$$\sigma^{(if)} = \frac{k_f}{k_i} \mathrm{Im} \sum_{n=1}^{3} \left[\frac{c_n^{(if)}}{z - z_n} - \frac{c_n^{(if)*}}{z + z_n^*} \right]$$

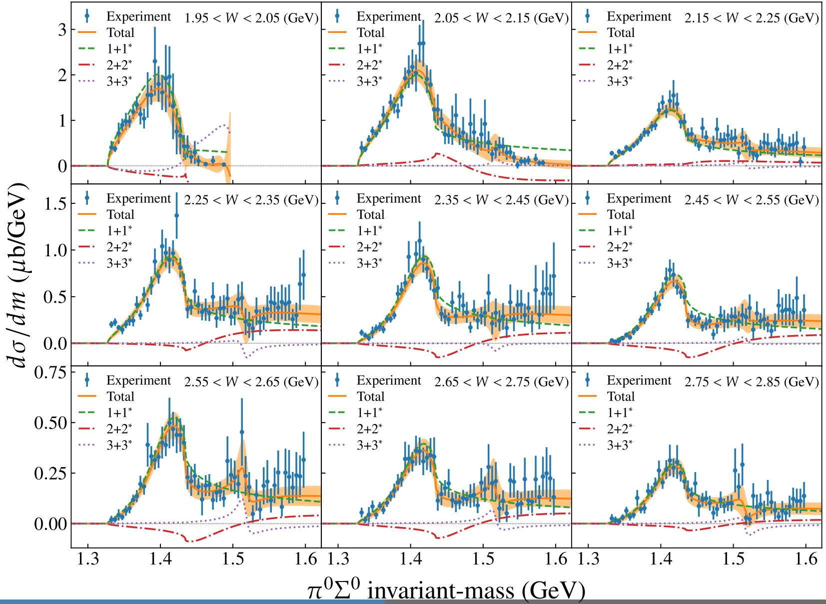


Λ(1405)

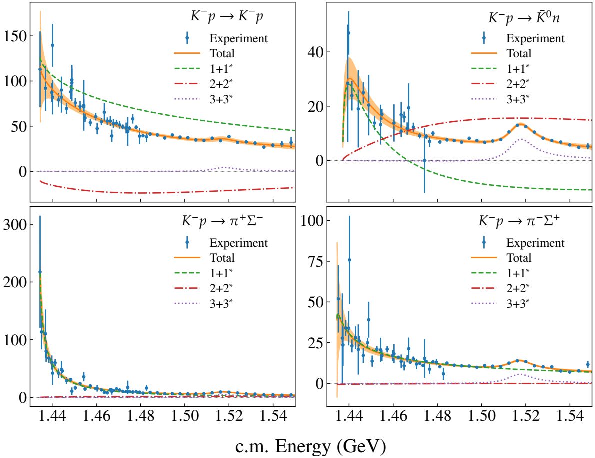


2.55 < W < 2.65 (GeV)
Experiment
Total
1+1*
2+2*
3+3*
dσ/dm (μb/GeV)
0.0
0.2
0.4
1.3
1.4
1.5
1.6
π+Σ− invariant-mass (GeV)

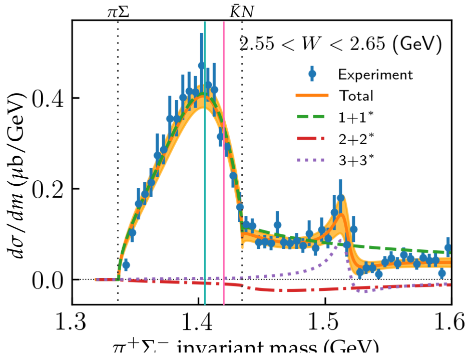
$\chi^2_{/dof}$ = 1.18

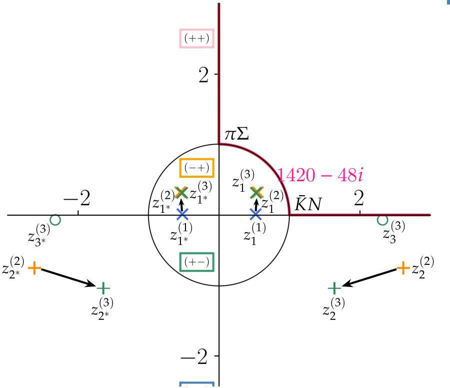

Experiment
Total
1+1*
2+2*
3+3*
1.95 < W < 2.05 (GeV)
2.05 < W < 2.15 (GeV)
2.15 < W < 2.25 (GeV)
2.25 < W < 2.35 (GeV)
2.35 < W < 2.45 (GeV)
2.45 < W < 2.55 (GeV)
2.55 < W < 2.65 (GeV)
2.65 < W < 2.75 (GeV)
2.75 < W < 2.85 (GeV)
dσ/dm (μb/GeV)
π⁰Σ⁰ invariant-mass (GeV)
1.3
1.4
1.5
1.6
0
1
2
3
0.0
0.5
1.0
1.5
0.00
0.25
0.50
0.75

Λ(1405)



$K^-p \to K^-p$
$K^-p \to \bar{K}^0 n$
$K^-p \to \pi^+\Sigma^-$
$K^-p \to \pi^-\Sigma^+$
Experiment
Total
1+1*
2+2*
3+3*
c.m. Energy (GeV)





	Pole 1	Pole 2	Pole 3
$z_n^{(3)}$	$0.5243{+}0.3159i \pm 0.0062 \pm 0.0058i$	$1.6402{-}1.042i \pm 0.0684 \pm 0.0904i$	$2.3227 - 0.0687i \pm 0.0033 \pm 0.0031i$
$\sqrt{s_n}^{(3)}$	$1.4203{-}0.0475i \pm 0.0011 \pm 0.0015i$	$1.4283{-}0.074i \pm 0.01 \pm 0.0037i$	$1.5138 - 0.0068i \pm 0.0003 \pm 0.0003i$

Chiral unitary calculation

Y. Ikeda, T. Hyodo and W. Weise, Phys. Lett. B 706, 63 (2011)

Y. Ikeda, T. Hyodo and W. Weise, Nucl. Phys. A 881, 98 (2012)

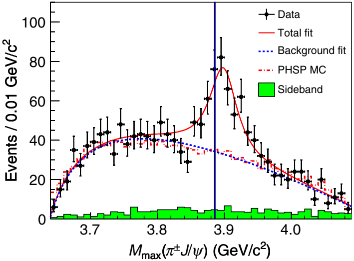
Z.-H. Guo and J. Oller, Phys. Rev. C 87, 3, 035202 (2013)

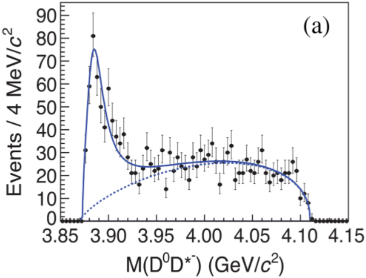
M. Mai and U.-G. Meißner, Eur. Phys. J. A 51, 3, 30 (2015)

approach	pole 1 [MeV]	pole 2 [MeV]
Refs. [14, 15], NLO	$1424^{+7}_{-23} - i\ 26^{+3}_{-14}$	$1381^{+18}_{-6} - i\ 81^{+19}_{-8}$
Ref. [17], Fit II	$1421^{+3}_{-2} - i\ 19^{+8}_{-5}$	$1388^{+9}_{-9} - i\ 114^{+24}_{-25}$
Ref. [18], solution #2	$1434^{+2}_{-2} - i\ 10^{+2}_{-1}$	$1330^{+4}_{-5} - i\ 56^{+17}_{-11}$
Ref. [18], solution #4	$1429^{+8}_{-7} - i\ 12^{+2}_{-3}$	$1325^{+15}_{-15} - i\ 90^{+12}_{-18}$

$Z(3900)$

M. Ablikim et al., Phys. Rev. Lett. 110, 252001 (BESIII)
M. Ablikim et al., Phys. Rev. Lett. 112, 022001 (BESIII)
Z. Q. Liu et al., Phys. Rev. Lett. 110, 252002 (Belle)





HALQCD: $\pi J/\psi$-$\rho\eta_c$-$D\bar{D}^*$, s-wave, (2+1)-flavor, m_π =410-700 MeV

Y. Ikeda, et.al., Phys. Rev. Lett. 117, 242001 (2016)

Case	[ttb]	[tbb]	[btb]
I	−146(112)(108) − i 38 (148)(32) −93(55)(21) − i 9(25)(7)	−177(116)(61) − i 175 (30)(22)	−369(129)(102) − i 207 (61)(20)
II	−102(84)(45) − i 14(11)(7) −59(67)(11) − i 3(12)(1)	−141(92)(64) − i 151 (149)(132)	−322(141)(111) − i 114 (96)(75)
III	−100(48)(29) −i 7(37)(17) −53(30)(5) −i 2(11)(3)	−127(52)(43) −i 199 (44)(28)	−356(108)(28) −i 277 (138)(95)

- HALQCD poles on the $\pi J/\psi$-$\bar{D}D^*$ sphere (m_π = 411 MeV)

$$e_p = (s_p - e_1^2)/(e_2^2 - e_1^2),$$
$$z_p = e_p^{1/2} + (e_p - 1)^{1/2}.$$
e_1 = 3508 MeV ($\pi J/\psi$), e_2 = 3959 MeV ($\bar{D}D^*$)

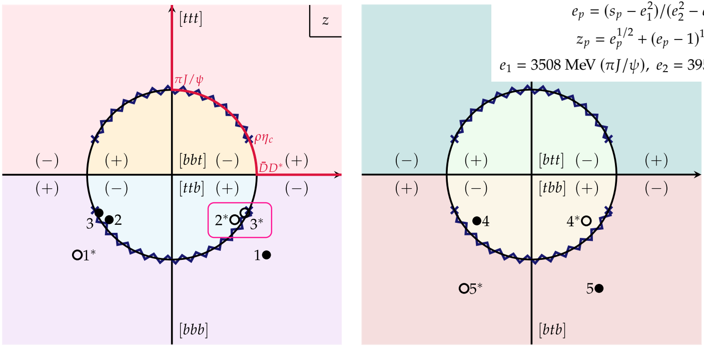


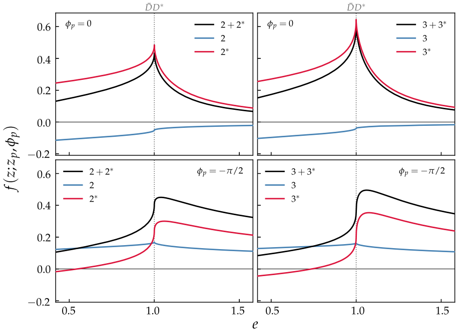
TABLE II. The uniformization variables, z_p, and the scaled energy, e_p, for S-matrix poles, 1–5 (Ime_p < 0), given in Ref. [20], and for their conjugate poles, 1*–5* (Ime_p > 0), not given in Ref. [20]. Also shown is the sheet on which each pole is positioned.

	$1, 1^*$	$2, 2^*$	$3, 3^*$	$4, 4^*$	$5, 5^*$
z_p	$\pm 1.11 - 0.95i$	$\mp 0.74 - 0.53i$	$\mp 0.86 - 0.45i$	$\mp 0.65 - 0.54i$	$\pm 0.79 - 1.34i$
e_p	$0.60 \mp 0.41i$	$0.66 \mp 0.09i$	$0.79 \mp 0.02i$	$0.60 \mp 0.17i$	$0.16 \mp 0.44i$
Sheet	$[bbb]$	$[ttb]$	$[ttb]$	$[tbb]$	$[btb]$

$Z(3900)$

Wren A. Yamada, Osamu Morimatsu, Toru Sato, Koichi Yazaki Phys. Rev. D 105, 014034 (2022)

- 2-channel Mittag-Leffler Expansion ($\pi J/\psi$-$\bar{D}D^*$)

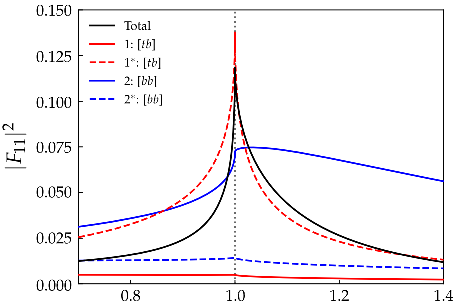


$$f(z;z_p,\phi_p) = -\frac{1}{\pi}\mathrm{Im}\left[\frac{\exp(i\phi_p)}{z - z_p} - \frac{\exp(-i\phi_p)}{z + z_p^*}\right]$$

- Separable potential model: $\pi J/\psi$-$\bar{D}D^*$ (HALQCD inspired)

$$V_{ij}(p', p) = g(p')v_{ij}g(p),$$

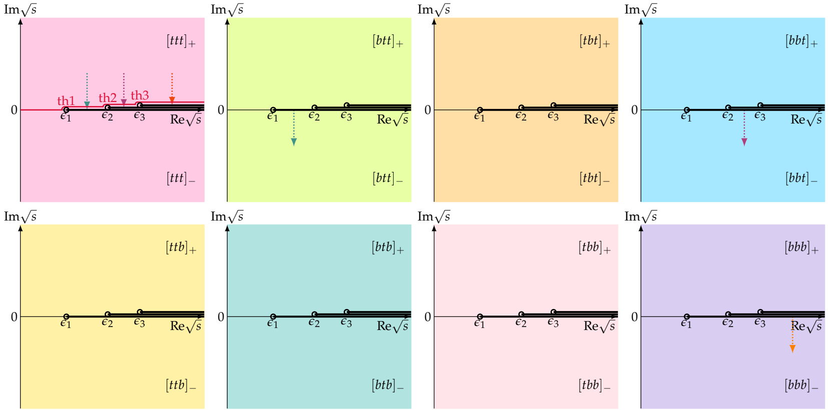
$$g(p) = \frac{\beta^2}{\beta^2 + p^2},\ v_{11} = v_{22} = 0,\ v_{12} = v$$



- Enhanced "threshold cusp" structure at $\bar{D}D^*$ threshold from poles 2^*, 3^* (pole on $[tb]_+$)

- **RS of the 3-channel S-matrix:** 2-body, RH cuts and poles
  - 2^3=8-sheeted $\sqrt{s}$-plane [ttt], [btt], [tbt], [bbt], [ttb], [bbb], [tbb], [btb]

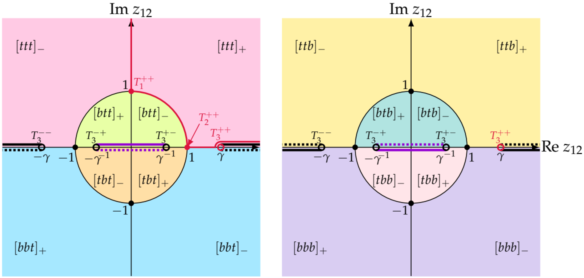
    e.g. $[ttb]_+$ means $Im\, q_1 > 0$, $Im\, q_2 > 0$, $Im\, q_3 < 0$ and $Im\sqrt{s} > 0$



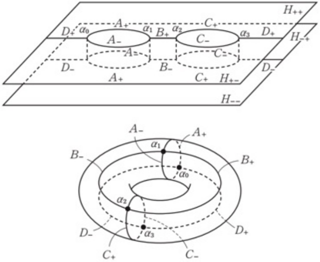
Analytic Structure of the RS of 3-channeled S-matrix

2-sheeted z_{12}-plane (z-plane using channel mass ϵ_1, ϵ_2)

$$q_1 = \frac{\Delta_{12}}{2}\left[z_{12} + 1/z_{12}\right], \quad q_2 = \frac{\Delta_{12}}{2}\left[z_{12} - 1/z_{12}\right], \quad q_3 = \frac{\Delta_{12}}{2z_{12}} \underbrace{\sqrt{(1 - z_{12}^2\gamma^2)(1 - z_{12}^2/\gamma^2)}}_{\text{sq.root cut } z_{12}=\pm\gamma,\pm 1/\gamma}, \quad \left(\gamma = \frac{\sqrt{\epsilon_3^2 - \epsilon_1^2} + \sqrt{\epsilon_3^2 - \epsilon_2^2}}{\Delta_{12}}\right)$$



W.Y. O.M. T.S. arXiv:2203.17069 [hep-ph], Fig.1



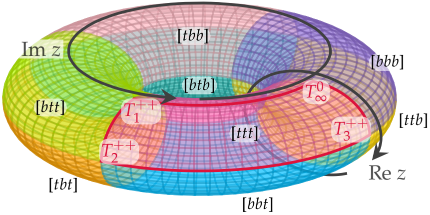
(楕円積分と楕円関数 おとぎの国の歩き方)

- 3-channel S-matrix has the structure of a Torus, fundamentally different from the 2-channel case (Riemann Sphere)!

H. Cohn, Conformal mapping on Riemann surfaces (Courier Corporation, 2014)
H. A. Weidenmüller Ann. Phys. (N.Y.) 28, 60 (1964)
R. G. Newton, Scattering Theory of Waves and Particles (Springer, 1982)

Torus representation of the 3-channel S-matrix

W. Yamada, O. Morimatsu, T. Sato, Phys. Rev. Lett. 129, 192001 (2022)



- 3-channel Uniformized variable: z

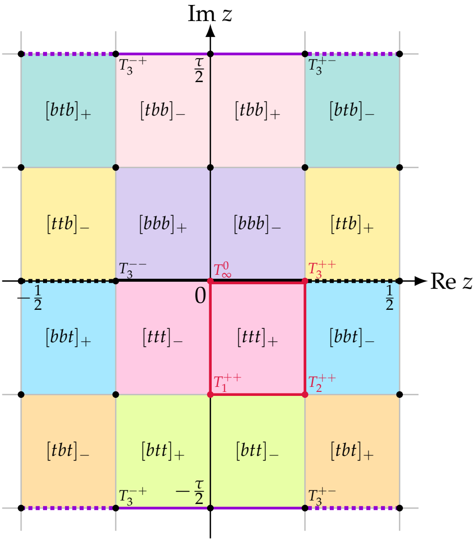
$$z[\tau] = \frac{1}{4K(k)} \underbrace{\int_0^{\gamma/z_{12}} \frac{d\xi}{\sqrt{1-\xi^2}\sqrt{1-k^2\xi^2}}}_{\text{elliptic integral}} .$$

$$k = \frac{1}{\gamma^2},\ K(k) = \int_0^{\pi/2} \frac{d\theta}{\sqrt{1-k^2\sin^2\theta}},\ \tau = \frac{K(\sqrt{1-k^2})}{2K(k)}$$

$$\gamma = \left(\sqrt{\epsilon_3^2-\epsilon_1^2} + \sqrt{\epsilon_3^2-\epsilon_2^2}\right)/\sqrt{\epsilon_2^2-\epsilon_1^2}$$

  - Double-periodicity

$$2\omega_1 = 1,\quad 2\omega_2 = \tau$$



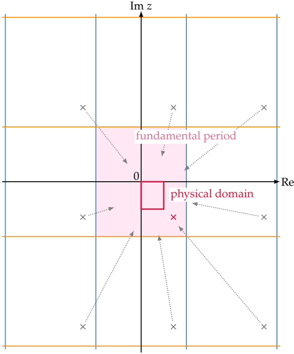
Double Periodicity

- Naive pole expansion + 1st, 2nd subtraction terms

$$A(z) = \sum_{z_i \in \Lambda^*} \left[\frac{r_i}{z - z_i} + \boxed{\sum_{m,n \neq 0} \frac{r_i}{z - z_i - \Omega_{mn}}} \right] + \text{(subtractions)}$$

Λ^* : fundamental period parallelogram, Ω_{mn} : lattice points

$$= C_0 + C_1 z + \sum_{z_i \in \Lambda^*} r_i \, \zeta(z - z_i)$$



Weierstrass Zeta function

$$\zeta(z) = \frac{1}{z} + \sum_{m,n \neq 0} \left[\frac{1}{z - \Omega_{mn}} + \frac{1}{\Omega_{mn}} + \frac{z}{\Omega_{mn}^2} \right] = \frac{1}{z} + \sum_{m,n \neq 0} \left[\frac{z^2}{(z - \Omega_{mn})\, \Omega_{mn}^2} \right]$$

- Boundary condition: $A \to 0$ at infinite energy

$$C_0 = - \sum_{z_i \in \Lambda^*} r_i \, \zeta(-z_i)$$

Mittag-Leffler Expansion under the Torus representation

$$A(z) = \sum_{z_i \in \Lambda^*} \boxed{r_i \left[\zeta(z - z_i) + \zeta(z_i) \right]}, \qquad \sum_{z_i \in \Lambda^*} r_i = \frac{1}{2\pi i} \oint_{\partial \Lambda^*} dz \, A(z) = 0$$

pole term

Mittag-Leffler Expansion on the Torus

Mittag-Leffler Expansion on the Torus with periods $(1, \tau)$

$$A(z) = \sum_{z_i \in \Lambda^*} r_i \left[\zeta(z - z_i; \tau) + \zeta(z_i; \tau) \right]$$

pole term, τ dependence?

Torus does not have one-to-one correspondence with τ
modular group SL(2,Z) induces an equivalent class of τ representing the same torus

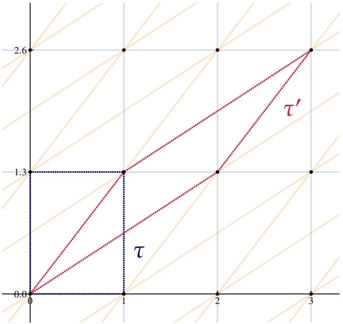
$$\begin{bmatrix} \omega_1 \\ \omega_2 \end{bmatrix} \mapsto \begin{bmatrix} \omega_1' \\ \omega_2' \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} \omega_1 \\ \omega_2 \end{bmatrix}, \quad a, b, c, d \in \mathbb{Z}, \quad det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = 1$$

$$z \mapsto z' = (c\tau + d)^{-1} z$$

$$\zeta(z;\tau) = \frac{1}{z} - \sum_{k=1}^{\infty} \mathcal{G}_{2k+2}(\tau) z^{2k+1} \mapsto \zeta(z;\tau') = (c\tau + d)\zeta(z;\tau)$$

$$\mathcal{G}_{2k+2} : \text{ Eisenstein series with weight } 2k+2$$

$$r_i^{[\tau]} \left[\zeta(z - z_i; \tau) + \zeta(z_i; \tau) \right] \mapsto \frac{r_i^{[\tau']}}{c\tau + d} (c\tau + d) \left[\zeta(z' - z_i'; \tau') + \zeta(z_i'; \tau') \right]$$
$$= r_i^{[\tau']} \left[\zeta(z' - z_i'; \tau') + \zeta(z_i'; \tau') \right]$$



- Pole decomposition of MLE is independent of the choice of τ

Model: ΛΛ-NΞ-ΣΣ

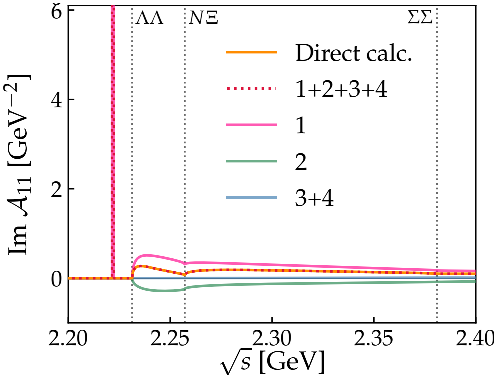
Model: ΛΛ-NΞ-ΣΣ [I = 0, J^P = 0$^+$, Flavor singlet]

D.B. Kaplan, M.J. Savage, M.B. Wise, Nucl.Phys. B478 (1996)

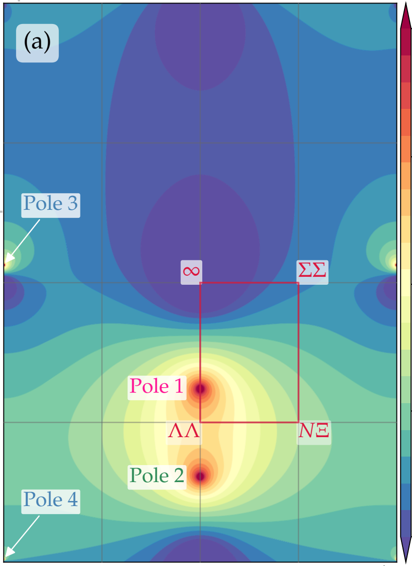
Y. Yamaguchi, T. Hyodo, Phys. Rev. C 94, 065207 (2016)

$$\mathcal{L}_{int} = -\frac{1}{2}[BB]^\dagger \hat{C}[BB], \quad \hat{C} = \frac{C}{8}\begin{bmatrix} 1 & 2 & -\sqrt{3} \\ 2 & 4 & -2\sqrt{3} \\ -\sqrt{3} & -2\sqrt{3} & 3 \end{bmatrix}$$

$$i\hat{A} = -i\hat{C}\left[1 - \hat{G}\hat{C}\right]^{-1}, \quad G_i = -i\mu_i k_i/2\pi$$



C = 40.00 [GeV^{-2}]



Model: ΛΛ-NΞ-ΣΣ

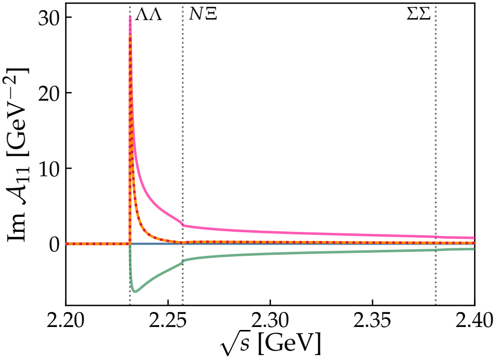
Model: ΛΛ-NΞ-ΣΣ [I = 0, J^P = 0^+, Flavor singlet]

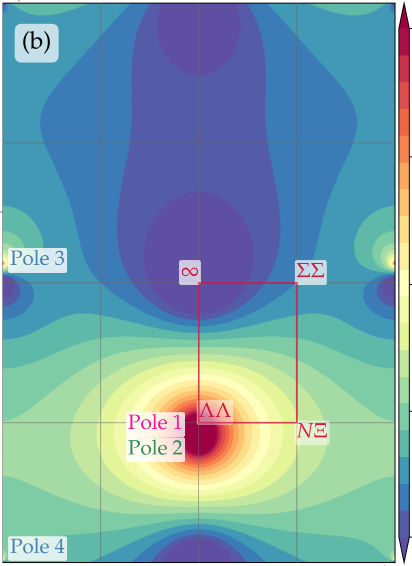
D.B. Kaplan, M.J. Savage, M.B. Wise, Nucl.Phys. B478 (1996)
Y. Yamaguchi, T. Hyodo, Phys. Rev. C 94, 065207 (2016)

$$\mathcal{L}_{int} = -\frac{1}{2}[BB]^\dagger \hat{C}[BB], \quad \hat{C} = \frac{C}{8}\begin{bmatrix} 1 & 2 & -\sqrt{3} \\ 2 & 4 & -2\sqrt{3} \\ -\sqrt{3} & -2\sqrt{3} & 3 \end{bmatrix}$$

$$i\hat{A} = -i\hat{C}\left[1 - \hat{G}\hat{C}\right]^{-1}, \quad G_i = -i\mu_i k_i/2\pi$$

C = 45.60 [GeV^{-2}]





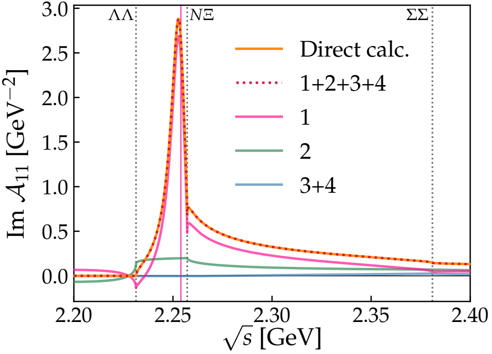
Model: ΛΛ-NΞ-ΣΣ

Model: ΛΛ-NΞ-ΣΣ [I = 0, J^P = 0$^+$, Flavor singlet]

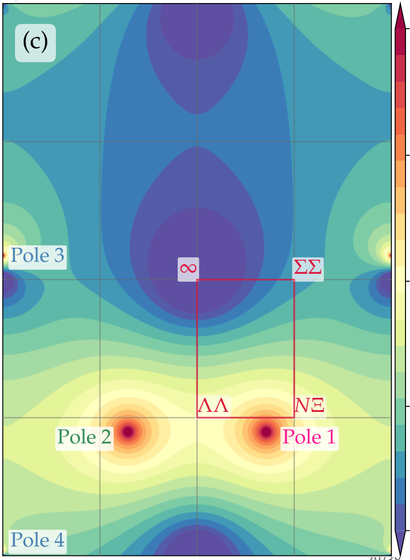
D.B. Kaplan, M.J. Savage, M.B. Wise, Nucl.Phys. B478 (1996)
Y. Yamaguchi, T. Hyodo, Phys. Rev. C 94, 065207 (2016)

$$\mathcal{L}_{int} = -\frac{1}{2}[BB]^\dagger \hat{C}[BB], \quad \hat{C} = \frac{C}{8}\begin{bmatrix} 1 & 2 & -\sqrt{3} \\ 2 & 4 & -2\sqrt{3} \\ -\sqrt{3} & -2\sqrt{3} & 3 \end{bmatrix}$$

$$i\hat{A} = -i\hat{C}\left[1 - \hat{G}\hat{C}\right]^{-1}, \quad G_i = -i\mu_i k_i/2\pi$$



C = 60.00 [GeV^{-2}]



Model: ΛΛ-NΞ-ΣΣ

Model: ΛΛ-NΞ-ΣΣ [I = 0, J^P = 0^+, Flavor singlet]

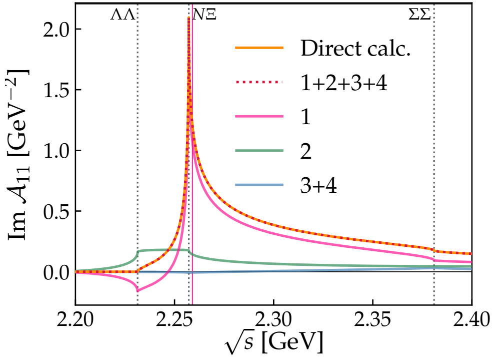
D.B. Kaplan, M.J. Savage, M.B. Wise, Nucl.Phys. B478 (1996)

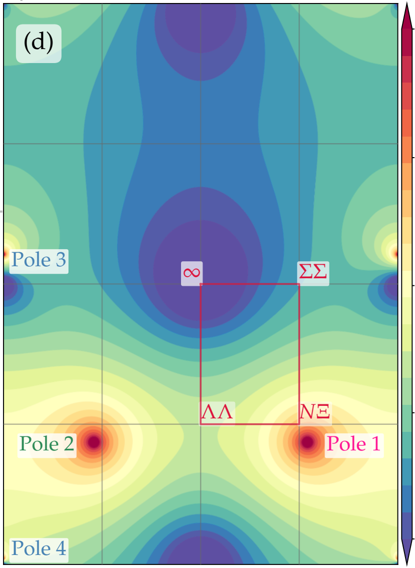
Y. Yamaguchi, T. Hyodo, Phys. Rev. C 94, 065207 (2016)

$$\mathcal{L}_{int} = -\frac{1}{2}[BB]^\dagger \hat{C}[BB], \quad \hat{C} = \frac{C}{8}\begin{bmatrix} 1 & 2 & -\sqrt{3} \\ 2 & 4 & -2\sqrt{3} \\ -\sqrt{3} & -2\sqrt{3} & 3 \end{bmatrix}$$

$$i\hat{A} = -i\hat{C}\left[1 - \hat{G}\hat{C}\right]^{-1}, \quad G_i = -i\mu_i k_i/2\pi$$

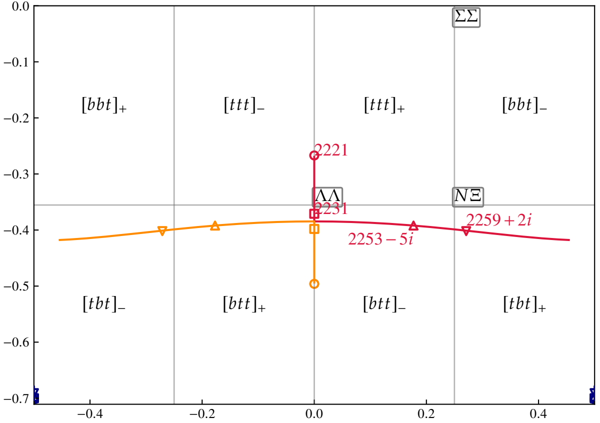
$C = 80.00\ [\mathrm{GeV}^{-2}]$





Model: ΛΛ-NΞ-ΣΣ

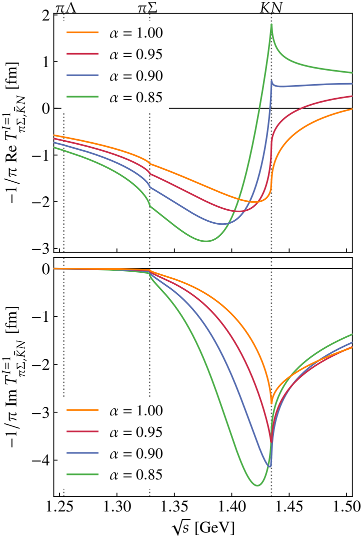
Pole Trajectory of Pole 1 & Pole 2 on the ΛΛ-NΞ-ΣΣ Torus

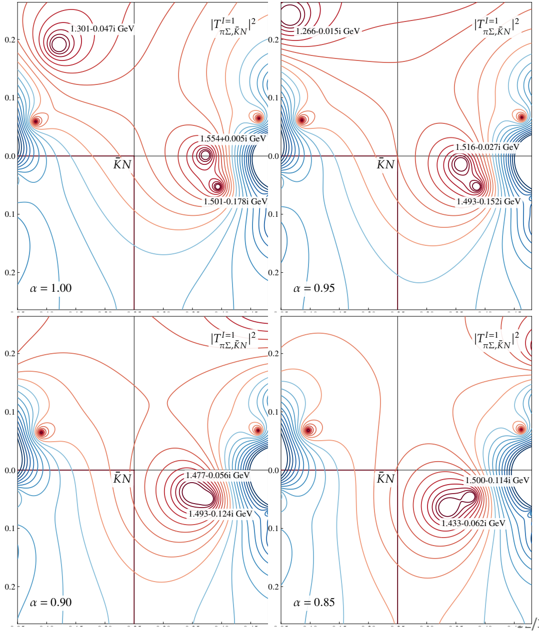


- Smooth transition of pole position and peak structure:
  Especially a smooth transition from a resonance pole on $[btt]_-$ to pole with positive imaginary complex energy on $[tbt]_+$ manifested as a 'cusp-like' shape
- 'Peak position' and 'width': closest physical point, distance on torus ≠ Re E_p, Im E_p

Model: $I = 1$ $\pi\Lambda$-$\pi\Sigma$-$\bar{K}N$

- **Chiral-Unitary Model LO:** $I = 1$ $\pi\Lambda$-$\pi\Sigma$-$\bar{K}N$

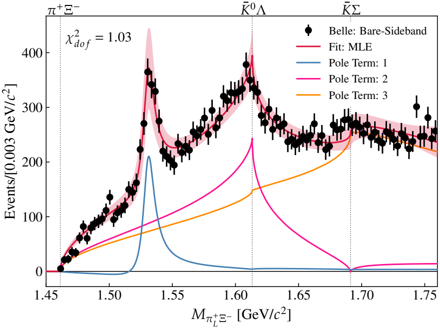


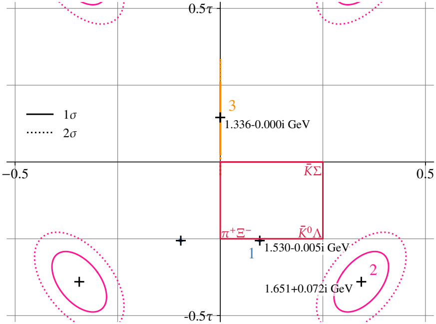


Ξ MLE Fit (preliminary)

- $\Xi_c^+ \to \Xi^-\pi^+\pi^+$ Belle, M. Sumihama et. al., PRL 122, 072501 (2019)
- Torus of 3-channel system: $\pi^+\Xi^-$, $\bar{K}^0\Lambda$, $\bar{K}\Sigma$
- 3-pole Mittag-Leffler Expansion

$$A(z) \approx \sum_{n=1}^{3} r_n \left[(\zeta(z - z_n) + \zeta(z_n))\right] + r_n^* \left[\zeta(z - z_n^*) - \zeta(z_n^*)\right]$$



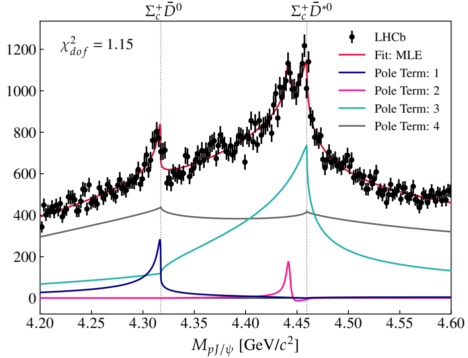


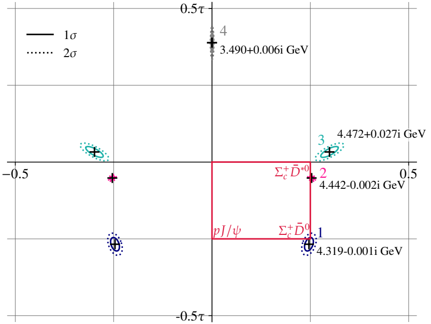
	Pole # 1	Pole # 2	Pole # 3
z	$0.096 - 0.223i \pm 0.003 \pm 0.002i$	$0.34 - 0.34i \pm 0.07 \pm 0.08i$	$0.00004 + 0.1i \pm 0.00003 \pm 0.1i$
ε_p [GeV]	$1.530 - 0.005i \pm 0.002 \pm 0.002i$	$1.7 + 0.07i \pm 0.1 \pm 0.08i$	$1.3 - 0.0001i \pm 0.4 \pm 0.0004i$
r_p^z [GeV^{-1}]	$-1.0 + 0.4i \pm 0.4 \pm 0.3i$	$-20 - 50i \pm 60 \pm 20i$	$-90000 + 50i \pm 50000 \pm 20i$
r_p^ε	$-1.1 + 0.4i \pm 0.4 \pm 0.4i$	$60 - 20i \pm 20 \pm 70i$	$-600 + 30000i \pm 500 \pm 20000i$

P_c MLE Fit (preliminary)

- $\Lambda_b^0 \to pJ/\psi K^-$ LHCb, R. Aaij et al. Phys. Rev. Lett. 122, 222001 (2019)
- Torus of 3-channel system: pJ/ψ, $\Sigma_c^+\bar{D}^0$, $\Sigma_c^+\bar{D}^{*0}$
- 4-pole Mittag-Leffler Expansion

$$A(z) \approx \sum_{n=1}^{4} r_n \left[(\zeta(z - z_n) + \zeta(z_n)) \right] + r_n^* \left[\zeta(z - z_n^*) - \zeta(z_n^*) \right]$$

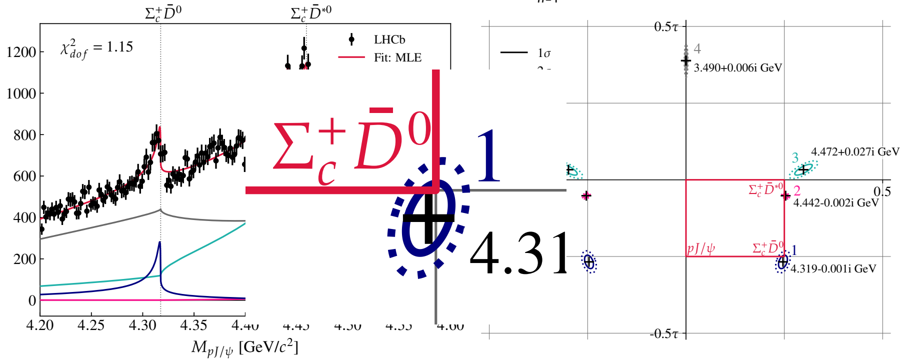




	Pole # 1	Pole # 2	Pole # 3	Pole # 4
z	$0.25 - 0.23i \pm 0.01 \pm 0.02i$	$0.253 - 0.045i \pm 0.006 \pm 0.006i$	$0.30 + 0.03i \pm 0.02 \pm 0.02i$	$0.000 + 0.34i \pm 0.005 \pm 0.03i$
ε_p [GeV]	$4.319 - 0.001i \pm 0.004 \pm 0.002i$	$4.442 - 0.002i \pm 0.004 \pm 0.004i$	$4.47 + 0.03i \pm 0.02 \pm 0.02i$	$3.5 + 0.01i \pm 0.5 \pm 0.08i$
r_p^z [GeV^{-1}]	$-2 - 5i \pm 4 \pm 4i$	$0.3 - 0.6i \pm 0.8 \pm 0.9i$	$50 - 20i \pm 20 \pm 10i$	$-80 + 120i \pm 2000 \pm 20i$
r_p^ε	$0.8 - 0.8i \pm 0.8 \pm 0.8i$	$-0.4 - 0.3i \pm 0.6 \pm 0.6i$	$60 + 20i \pm 20 \pm 20i$	$-2100 - 1000i \pm 600 \pm 30000i$

- $\Lambda_b^0 \to pJ/\psi K^-$ LHCb, R. Aaij et al. Phys. Rev. Lett. 122, 222001 (2019)
- Torus of 3-channel system: pJ/ψ, $\Sigma_c^+\bar{D}^0$, $\Sigma_c^+\bar{D}^{*0}$
- 4-pole Mittag-Leffler Expansion

$$A(z) \approx \sum_{n=1}^{4} r_n \left[(\zeta(z - z_n) + \zeta(z_n))\right] + r_n^* \left[\zeta(z - z_n^*) - \zeta(z_n^*)\right]$$



	Pole # 1	Pole # 2	Pole # 3	Pole # 4
z	$0.25 - 0.23i \pm 0.01 \pm 0.02i$	$0.253 - 0.045i \pm 0.006 \pm 0.006i$	$0.30 + 0.03i \pm 0.02 \pm 0.02i$	$0.000 + 0.34i \pm 0.005 \pm 0.03i$
ϵ_p [GeV]	$4.319 - 0.001i \pm 0.004 \pm 0.002i$	$4.442 - 0.002i \pm 0.004 \pm 0.004i$	$4.47 + 0.03i \pm 0.02 \pm 0.02i$	$3.5 + 0.01i \pm 0.5 \pm 0.08i$
r_p^z [GeV^{-1}]	$-2 - 5i \pm 4 \pm 4i$	$0.3 - 0.6i \pm 0.8 \pm 0.9i$	$50 - 20i \pm 20 \pm 10i$	$-80 + 120i \pm 2000 \pm 20i$
r_p^ϵ	$0.8 - 0.8i \pm 0.8 \pm 0.8i$	$-0.4 - 0.3i \pm 0.6 \pm 0.6i$	$60 + 20i \pm 20 \pm 20i$	$-2100 - 1000i \pm 600 \pm 30000i$

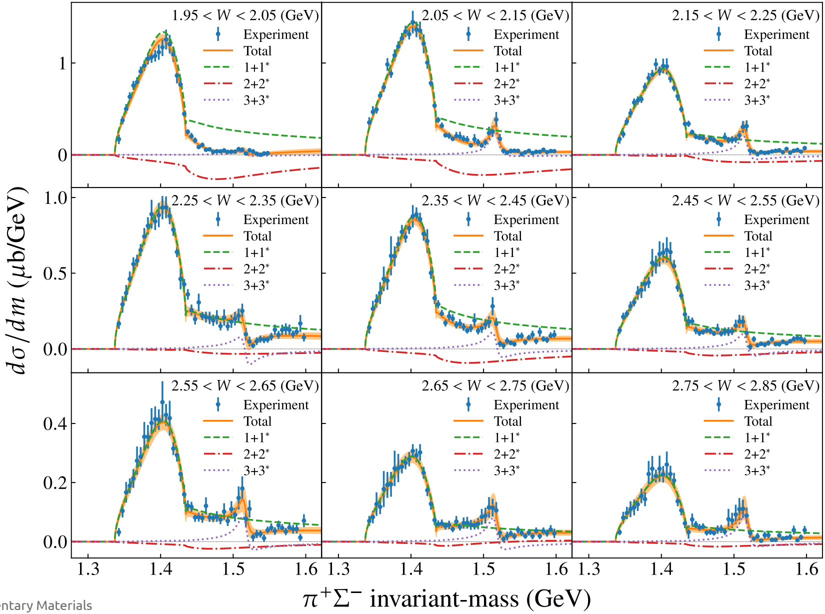
Summary

- Non-trivial analytic structure of S-matrix in energy near the thresholds
  Breit-Wigner does not reflect the proper structure
- Uniformization: clarification pole position $\leftrightarrow$ spectrum
  2-channel S-matrix: Sphere, 3-channel S-matrix: Torus
- Mittag-Leffler Expansion
  Pole Expansion accounting the non-trivial analytic structure of S-matrix
  For 3-channel case, double-periodicity of torus has to be considered
- Line shapes: Enhanced structure in spectrum $\rightarrow$ Existence of nearby poles
  - Smooth transition of peak structure (under smooth transition of pole)
  - 'Resonances' ($[bt(t)]_-$,$[bb(t)]_-$,$[bb(b)]_-$),
    'Cusp'-shaped enhancements ($[tb(t)]_+$,$[(t)tb]_+$)
  - Peak position ≈ closest physical point on uniformized plane, $\neq \mathrm{Re}\, E_{pole}$
- Application of Mittag-Leffler Expansion:
  - $\Lambda(1405)$: Primary pole on $[bt]$-sheet, $E_p > 1420 > 1405$ MeV
  - $Z(3900)$: Possible contribution from poles on $[(t)tb]_+$
  - 3-channel Mittag-Leffler Expansion to Ξ, P_c

Thank You!!

Supplementary Materials

Mittag-Leffler Expansion to Λ(1405): 3 pole-terms



Mittag-Leffler Expansion to Λ(1405): 3 pole-terms

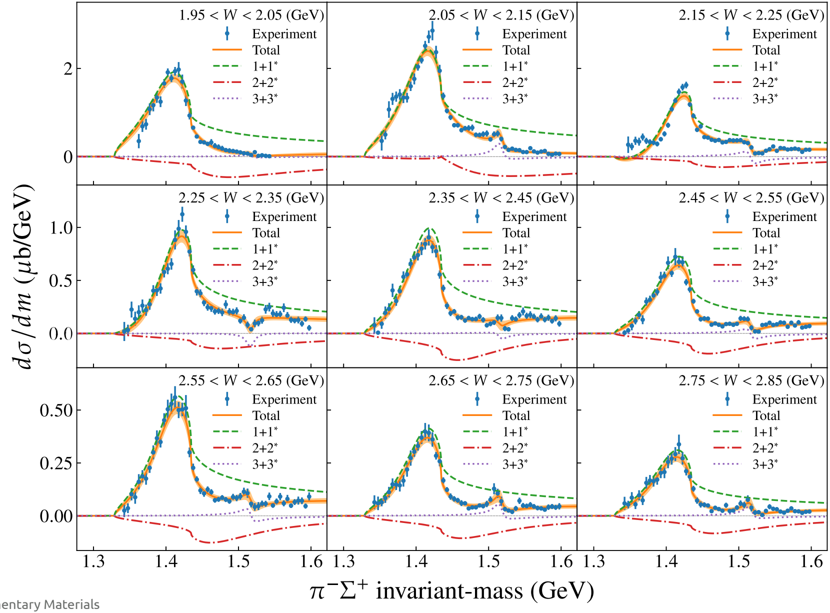


TABLE IV. Results for the residues of the invariant-mass distributions of $\pi^0\Sigma^0$ in units of μb/GeV in nine bins of center-of-mass energy W by the uniformized Mittag-Leffler expansion with $m = 3$.

W (GeV)	Pole 1	Pole 2	Pole 3
1.95–2.05	$-0.6515 + 0.3471i \pm 0.2256 \pm 0.1211i$	$0.5316 - 1.2492i \pm 0.7596 \pm 1.3581i$	$1.3537 - 0.6183i \pm 2.7107 \pm 1.0427i$
2.05–2.15	$-0.3179 + 0.5296i \pm 0.0374 \pm 0.06i$	$-0.3174 - 0.6043i \pm 0.1764 \pm 0.1197i$	$-0.011 + 0.0019i \pm 0.0104 \pm 0.0121i$
2.15–2.25	$-0.1085 + 0.3535i \pm 0.0209 \pm 0.0333i$	$-0.0763 + 0.0737i \pm 0.1051 \pm 0.0997i$	$-0.0015 - 0.009i \pm 0.0108 \pm 0.0099i$
2.25–2.35	$-0.053 + 0.2798i \pm 0.0154 \pm 0.0245i$	$0.0799 + 0.2387i \pm 0.0854 \pm 0.0871i$	$0.0081 - 0.0087i \pm 0.0086 \pm 0.0082i$
2.35–2.45	$0.0027 + 0.2895i \pm 0.0139 \pm 0.0227i$	$0.1853 + 0.2406i \pm 0.0828 \pm 0.0885i$	$0.0052 - 0.001i \pm 0.0079 \pm 0.0073i$
2.45–2.55	$0.0223 + 0.2323i \pm 0.0097 \pm 0.0164i$	$0.1871 + 0.2054i \pm 0.0618 \pm 0.0691i$	$-0.0038 - 0.0032i \pm 0.0061 \pm 0.0063i$
2.55–2.65	$0.0088 + 0.1641i \pm 0.0084 \pm 0.0141i$	$0.1101 + 0.1044i \pm 0.0479 \pm 0.0491i$	$-0.0051 - 0.0098i \pm 0.0054 \pm 0.0042i$
2.65–2.75	$-0.0018 + 0.1221i \pm 0.0076 \pm 0.0126i$	$0.0883 + 0.1107i \pm 0.0414 \pm 0.0428i$	$-0.0026 - 0.0058i \pm 0.0047 \pm 0.0038i$
2.75–2.85	$0.0089 + 0.094i \pm 0.0058 \pm 0.009i$	$0.0417 + 0.0439i \pm 0.0317 \pm 0.0294i$	$0.0018 - 0.0052i \pm 0.0025 \pm 0.0032i$

Residues $\pi^+\Sigma^-$: W. Yamada, O. Morimatsu, PRC 103, 045201

TABLE II. Results for the residues of the invariant-mass distributions of $\pi^+\Sigma^-$ in units of μb/GeV in nine bins of center-of-mass energy W by the uniformized Mittag-Leffler expansion with $m = 3$.

W (GeV)	Pole 1	Pole 2	Pole 3
1.95–2.05	$-0.3486 + 0.3026i \pm 0.0154 \pm 0.0149i$	$0.2487 - 0.122i \pm 0.053 \pm 0.0342i$	$-0.0016 - 0.0029i \pm 0.0013 \pm 0.0014i$
2.05–2.15	$-0.3809 + 0.3245i \pm 0.0156 \pm 0.0135i$	$0.1451 - 0.1877i \pm 0.0442 \pm 0.0225i$	$-0.0175 - 0.0081i \pm 0.0034 \pm 0.0023i$
2.15–2.25	$-0.2662 + 0.1989i \pm 0.0121 \pm 0.0096i$	$0.0294 - 0.0919i \pm 0.028 \pm 0.0183i$	$-0.0108 - 0.0133i \pm 0.0029 \pm 0.0021i$
2.25–2.35	$-0.2539 + 0.208i \pm 0.013 \pm 0.0106i$	$0.0165 - 0.0339i \pm 0.0318 \pm 0.0227i$	$0.0014 - 0.0122i \pm 0.0023 \pm 0.0021i$
2.35–2.45	$-0.2016 + 0.2142i \pm 0.0131 \pm 0.0104i$	$0.0864 - 0.0442i \pm 0.0306 \pm 0.0189i$	$-0.004 - 0.0105i \pm 0.0021 \pm 0.0019i$
2.45–2.55	$-0.1595 + 0.1369i \pm 0.0097 \pm 0.008i$	$0.0423 - 0.0179i \pm 0.0219 \pm 0.0151i$	$-0.0038 - 0.0091i \pm 0.0018 \pm 0.0017i$
2.55–2.65	$-0.1072 + 0.0925i \pm 0.008 \pm 0.006i$	$0.025 - 0.0066i \pm 0.0169 \pm 0.0119i$	$-0.0043 - 0.0065i \pm 0.0016 \pm 0.0014i$
2.65–2.75	$-0.0891 + 0.057i \pm 0.0065 \pm 0.0046i$	$0.0189 + 0.0133i \pm 0.0139 \pm 0.01i$	$-0.0039 - 0.0062i \pm 0.0014 \pm 0.0012i$
2.75–2.85	$-0.0657 + 0.0466i \pm 0.0056 \pm 0.0042i$	$0.0161 - 0.0066i \pm 0.0115 \pm 0.008i$	$-0.0053 - 0.0051i \pm 0.0013 \pm 0.0011i$

Residues $\pi^-\Sigma^+$: W. Yamada, O. Morimatsu, PRC 103, 045201

TABLE III. Results for the residues of the invariant-mass distributions of $\pi^-\Sigma^+$ in units of μb/GeV in nine bins of center-of-mass energy W by the uniformized Mittag-Leffler expansion with $m = 3$.

W (GeV)	Pole 1	Pole 2	Pole 3
1.95–2.05	$-0.2247 + 0.542i \pm 0.0319 \pm 0.0262i$	$0.358 - 0.2978i \pm 0.0864 \pm 0.0491i$	$-0.0013 - 0.0038i \pm 0.0017 \pm 0.0017i$
2.05–2.15	$-0.1119 + 0.7353i \pm 0.035 \pm 0.0301i$	$0.0861 - 0.542i \pm 0.0823 \pm 0.0456i$	$-0.0165 - 0.0155i \pm 0.0035 \pm 0.0033i$
2.15–2.25	$0.1962 + 0.4702i \pm 0.02 \pm 0.0162i$	$0.2154 - 0.1012i \pm 0.0524 \pm 0.0325i$	$0.002 - 0.0171i \pm 0.0027 \pm 0.0026i$
2.25–2.35	$0.0662 + 0.3112i \pm 0.0144 \pm 0.0129i$	$0.1313 - 0.0568i \pm 0.0374 \pm 0.0233i$	$0.0081 + 0.001i \pm 0.0014 \pm 0.002i$
2.35–2.45	$-0.0017 + 0.3091i \pm 0.0116 \pm 0.0116i$	$0.2839 + 0.0335i \pm 0.0461 \pm 0.0327i$	$0.0028 - 0.0026i \pm 0.0018 \pm 0.0016i$
2.45–2.55	$-0.0119 + 0.2237i \pm 0.009 \pm 0.0088i$	$0.2132 + 0.017i \pm 0.0346 \pm 0.0236i$	$0.0004 - 0.006i \pm 0.0014 \pm 0.0012i$
2.55–2.65	$-0.0189 + 0.1726i \pm 0.0075 \pm 0.0073i$	$0.1377 - 0.0008i \pm 0.0248 \pm 0.0162i$	$-0.0006 - 0.0038i \pm 0.001 \pm 0.0011i$
2.65–2.75	$-0.0123 + 0.1263i \pm 0.0062 \pm 0.0055i$	$0.1136 - 0.0044i \pm 0.02 \pm 0.0131i$	$-0.0029 - 0.0035i \pm 0.001 \pm 0.0009i$
2.75–2.85	$-0.0173 + 0.0932i \pm 0.0055 \pm 0.005i$	$0.0859 - 0.0121i \pm 0.016 \pm 0.0096i$	$-0.0021 - 0.0028i \pm 0.0009 \pm 0.0007i$

Optical Theorem

$$\pi |T_X|^2 = \sum_{IJ} \langle\phi|\, F_I^\dagger G_I^\dagger (G_{I0}^{-1})^\dagger \,|X\rangle\, \mathrm{Im}\, G_0^X \langle X|\, G_{J0}^{-1} G_J F_J \,|\phi\rangle$$

$$= \sum_{IJ} \langle\phi|\, F_I^\dagger G_I^\dagger (G_{I0}^{-1})^\dagger P_X \mathrm{Im}\, G_0 P_X G_{J0}^{-1} G_J F_J \,|\phi\rangle \qquad P_X : \text{projection operator onto } |X\rangle$$

$$= \langle\phi| \sum_I (F_I^\dagger G_I^\dagger (G_{I0}^{-1})^\dagger) P_X \mathrm{Im}\, G_0 P_X \sum_J (G_{J0}^{-1} G_J F_J) \,|\phi\rangle$$

$$= \langle\phi|\, F^\dagger G^\dagger (G_0^{-1})^\dagger P_X \mathrm{Im}\, G_0 P_X G_0^{-1} G F \,|\phi\rangle \qquad G_0^{-1} G F \equiv \sum_I G_{I0}^{-1} G_I F_I$$

$$= \mathrm{Im}\, \langle\phi|\, F^\dagger G^\dagger (G_0^{-1})^\dagger P_X G_0 P_X G_0^{-1} G F \,|\phi\rangle$$

G : Green's operator, I, J : channel index

- $\langle\phi|\, F^\dagger G^\dagger (G_0^{-1})^\dagger P_X G_0 P_X G_0^{-1} G F \,|\phi\rangle$ inherits the analytic properties (not all) of the Green's function

Partialwave analysys

SO(3) → block diagnalizable

$$\hat{A} = \hat{A}_0 \oplus \hat{A}_1 \oplus \hat{A}_2 \cdots$$

$$A(k, \theta, \phi) = \sum_{l,m} A_l(k)\, Y_{lm}(\theta, \phi) Y^*_{lm}(\theta, \phi)$$

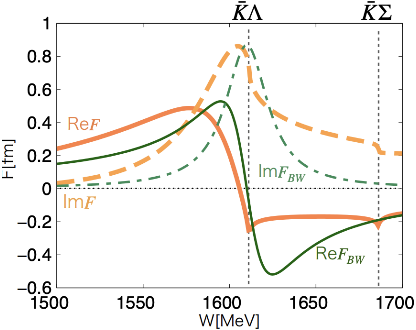
global structure of the RS of a_l: same → ML-Expansion

$$A_l = \frac{2l+1}{k \cot \delta_l - ik}, \quad k^{2l+1} \cot \delta_l = -\frac{1}{a_l} + \frac{1}{2} r_l k^2 + \cdots$$

Lineshape: Example

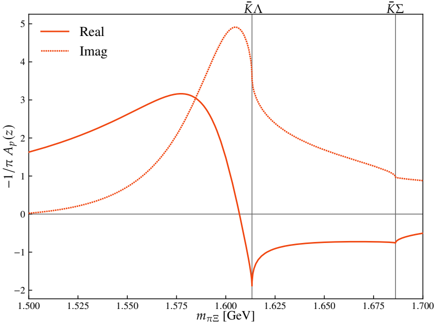
Model Calculation

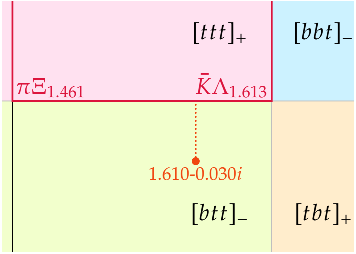
T. Nishibuchi, T. Hyodo, Contribution to HYP 2022, e-Print: 2208.14608 [hep-ph]



z_{th} = 1610 – 30i MeV

3-channel MLE 1-pole term: Arg[r_n]= −0.23π





Model: Pole Properties

W. Yamada, O. Morimatsu, T. Sato, Phys. Rev. Lett. 129, 192001 (2022)

TABLE I. Pole positions and residues of the $\Lambda\Lambda \to \Lambda\Lambda$ elastic scattering amplitude, $\mathcal{A}_{11}$, for cases (a)–(d). The first and second rows are the pole positions, z_i, and residues, r_i, respectively, on the torus. The third row is the complex center-of-mass energy of the pole, $\sqrt{s_i}$, in units of [GeV] and the complex Riemann sheet. The threshold energies, ε_1, ε_2, and ε_3, are 2.231, 2.257, and 2.381 GeV, respectively.

	C (GeV^{-2})	Pole 1	Pole 2	Pole 3	Pole 4
(a)	40.00	$-0.267i$ $0.172i$ 2.221 $[ttt]$	$-0.496i$ $-0.154i$ 2.200 $[btt]$	$0.5+0.043i$ $-0.015i$ $-1.802i$ $[ttb]$	$0.5-0.702i$ $-0.004i$ $13.477i$ $[tbt]$
(b)	45.60	$-0.371i$ $1.750i$ 2.231 $[btt]$	$-0.398i$ $-1.727i$ 2.229 $[btt]$	$0.5+0.048i$ $-0.018i$ $-1.252i$ $[ttb]$	$0.5-0.700i$ $-0.005i$ $11.722i$ $[tbt]$
(c)	60.00	$0.177-0.392i$ $-0.215+0.018i$ $2.253-0.005i$ $[btt]$	$-0.177-0.392i$ $0.215+0.018i$ $2.253+0.005i$ $[btt]$	$0.5+0.060i$ $-0.027i$ 0.907 $[ttb]$	$0.5-0.697i$ $-0.009i$ $8.657i$ $[tbt]$
(d)	80.00	$0.271-0.402i$ $-0.249+0.028i$ $2.259+0.002i$ $[tbt]$	$-0.271-0.402i$ $0.249+0.028i$ $2.259-0.002i$ $[tbt]$	$0.5+0.073i$ $-0.038i$ 1.510 $[ttb]$	$0.5-0.691i$ $-0.017i$ $6.124i$ $[tbt]$